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FRANK MORLEY, EDITOR

A. COHEN, ASSISTANT EDITOR

WITH THE COÖPERATION OF

CHARLOTTE A. SCOTT, A. B. COBLE

AND OTHER MATHEMATICIANS

PUBLISHED UNDER THE AUSPICES OF THE JOHNS HOPKINS UNIVERSITY

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MULTIPLE BINARY FORMS WITH THE CLOSURE PROPERTY.

By Arthur B. Coble.*

Introduction.

Interesting accounts of the many investigations which have centered about the Poncelet polygons associated with the double binary forms of order (2, 2) with the closure property have been given by G. Loria† and H. S. White.‡ The latter has called attention also to the existence of forms of order (3, 3) with a similar property.§

It is the purpose of this paper to discuss the conditions for closure of the general double binary form of order (k, κ) and to indicate certain possible extensions to multiple forms. It will appear that the closure property for a given form implies the existence of a *complementary* form such that the product of a given form and its complement can be expressed as a determinant for which the closure property is inherent. A number of forms with the closure property are constructed—a number sufficient to indicate that the theory here developed has a reasonable content.

The treatment in general is algebraic though geometric interpretations are constantly helpful. The striking application of the elliptic functions to the Poncelet case seems not to be capable of immediate extension to forms of higher degree since the conditions for closure can not be expressed by means of integrals of the first kind alone.

1. *Poristic Forms and their Complements.*—We begin with the double binary form of order (k, κ)

$$F = (\alpha t)^k (a\tau)^\kappa = 0 \tag{1}$$

and with one pair of values t_0, τ_0 which satisfies it. The value t_0 determines in (1) κ values of τ, say $\tau_0, \tau_1, \cdots$, each of which determines $k - 1$ values of t in addition to t_0 and these values of t determine in turn new values of τ. Proceeding in this way we obtain values $t_0, t_1, t_2, \cdots; \tau_0, \tau_1, \tau_2, \cdots$. If after a finite number of repetitions of this procedure we obtain a set of n values t and ν values τ such that each $t(\tau)$ determines in (1) κ (k) values of $\tau(t)$ which are already included in the set of $\nu(n)$ values of $\tau(t)$ then we

* Prepared under the auspices of the Carnegie Institution of Washington, D. C.

† "I polygoni di Poncelet," Torino (1889).

‡ "Poncelet polygons," *Science* (Feb. 4, 1916), pp. 149–158.

§ *Proceedings of the National Academy of Sciences*, Vol. 1 (Aug., 1915), p. 464; and Vol. 2 (June, 1916), p. 337.

shall say that F *admits the configuration* $\Delta_{n,\nu}^{k,\kappa}$, $t_0, t_1, t_2, \cdots, t_{n-1}$; $\tau_0, \tau_1, \tau_2, \cdots, \tau_{\nu-1}$. Indeed in any geometric interpretation of the form F the closed set just mentioned would give rise to a geometric configuration. This configuration would have moreover a type of regularity since each of the t's determines κ of the τ's and each of the τ's is determined by k of the t's. It is clear from the method of derivation of the closed set that the configuration is determined by any one of its *elements* t or of its *elements* τ.

The number of elements t which with elements τ, i.e., of *pairs* of elements t, τ, will satisfy the form F is $n\kappa$, or equally well νk, so that

$$n\kappa = \nu k. \tag{2}$$

If F admits one configuration $\Delta_{n,\nu}^{k,\kappa}$ it does not necessarily admit an infinite number. Classic examples of this are at hand. If however it does admit an infinite number the form F will be called *poristic* and the infinitude of configurations which it determines will be called a *porism*.

When a configuration $\Delta_{n,\nu}^{k,\kappa}$ satisfies a form F there is determined a *complementary configuration* $\Delta_{n,\nu}^{n-k,\nu-\kappa}$ made up of those pairs of elements t, τ of the given configuration which do *not* satisfy F.

If the form F is poristic the ∞^1 sets of n elements t which belong to the ∞^1 configurations $\Delta_{n,\nu}^{k,\kappa}$ are determined by a *pencil* of binary n-ics, $(\gamma t)^n + \lambda(\delta t)^n = 0$. For evidently a set of n elements t is determined by any element of the set. Similarly the ∞^1 sets of ν elements τ in the ∞^1 configurations are determined by a pencil of binary ν-ics, $(c\tau)^\nu + \mu(d\tau)^\nu = 0$. Since each set of n t's makes up with a unique set of ν τ's a configuration and conversely, these two pencils are projectively related, and for proper choice of $(c\tau)^\nu$, $(d\tau)^\nu$ in the second pencil we shall have $\mu = \lambda$. Eliminating λ between the two pencils we get

$$D_{n,\nu} = \begin{vmatrix} (\gamma t)^n & (\delta t)^n \\ (c\tau)^\nu & (d\tau)^\nu \end{vmatrix} = 0. \tag{3}$$

The form $D_{n,\nu}$ from its nature as a determinant is poristic. It determines ∞^1 *complete* configurations $\Delta_{n,\nu}^{n,\nu}$, i.e., sets of n t's and ν τ's such that each t determines *all* the τ's and vice versa. These configurations arise from the vanishing of the general column, $\lambda_1(\gamma t)^n + \lambda_2(\delta t)^n = 0$, $\lambda_1(c\tau)^\nu + \lambda_2(d\tau)^\nu = 0$, of the determinant. We shall call the form $D_{n,\nu}$ a *complete* poristic form and its configurations a *complete* porism. Clearly $D_{n,\nu}$ determined as above will contain the original poristic form as a factor and the complementary factor

$$F' = (\beta t)^{n-k}(b\tau)^{\nu-\kappa} \tag{4}$$

will also be poristic and will admit the configurations $\Delta_{n,\nu}^{n-k,\nu-\kappa}$ which are complementary to the configurations $\Delta_{n,\nu}^{k,\kappa}$ which satisfy F. Hence

(5) *Every poristic double binary form F determines a complementary poristic form F' such that each configuration of F coupled with a complementary configuration of F' makes up a configuration of the complete poristic form $FF' = D_{n,\nu}$.*

2. *Conditions for Closure. Corollaries.*—We shall now prove the theorem

(6) *The necessary and sufficient conditions that a given form $F_{k,\kappa}$ shall be poristic with configurations $\Delta^{k,\kappa}_{n,\nu}$ (where n, ν, k, κ are integers subject to the condition (2)) are (A) that the given form shall admit* ONE *such configuration, and (B) that there shall exist a form $F_{n-k,\nu-\kappa}$ which shall admit the complementary configuration.*

The necessity of these conditions appears in Sec. 1; we here prove them sufficient. Let the form F in (1) admit the $\Delta^{k,\kappa}_{n,\nu}$ with elements $t_0, t_1, \cdots, t_{n-1}$; $\tau_0, \tau_1, \cdots, \tau_{\nu-1}$. By hypothesis (B) the form F' in (4) exists such that in the product

$$FF' = (\alpha t)^k (a\tau)^k \cdot (\beta t)^{n-k} (b\tau)^{\nu-\kappa} = 0$$

each of the n t's determines all of the ν τ's and vice versa. Suppose that for $t = t_0, t_1, \cdots, t_{n-1}$ the product FF' takes the values $r_0(d\tau)^\nu, r_1(d\tau)^\nu, \cdots, r_{n-1}(d\tau)^\nu$ where $(d\tau)^\nu = 0$ has roots $\tau_0, \cdots, \tau_{\nu-1}$. We can determine (in any one of ∞^1 ways) a binary n-ic, $(\gamma t)^n$, which for $t = t_0, \cdots, t_{n-1}$ takes the values $r_0, \cdots, r_{n-1}$. Then the difference $FF' - (\gamma t)^n \cdot (d\tau)^\nu$ vanishes identically for $t = t_0, \cdots, t_{n-1}$ and therefore contains the factor $(\delta t)^n$, where $(\delta t)^n = 0$ has roots $t_0, \cdots, t_{n-1}$, and a complementary factor, say $-(c\tau)^\nu$. Hence

$$FF' = (\gamma t)^n \cdot (d\tau)^\nu - (\delta t)^n \cdot (c\tau)^\nu = \begin{vmatrix} (\gamma t)^n & (\delta t)^n \\ (c\tau)^\nu & (d\tau)^\nu \end{vmatrix}.$$

But as has been pointed out the determinant is necessarily poristic and admits ∞^1 configurations $\Delta^{n,\nu}_{n,\nu}$. Since it factors into FF' each configuration $\Delta^{n,\nu}_{n,\nu}$ breaks up into configurations $\Delta^{k,\kappa}_{n,\nu}$ and $\Delta^{n-k,\nu-\kappa}_{n,\nu}$ which satisfy respectively the forms F and F'. Hence each of the forms F and F' are poristic.

Even if the given form F admits one $\Delta^{k,\kappa}_{n,\nu}$ the number, $n(\nu - \kappa)$, of *apparent* conditions on an arbitrary form F' which ensure that it shall admit the complementary $\Delta^{n-k,\nu-\kappa}_{n,\nu}$ is more than F' with $(n-k+1)(\nu-\kappa+1)$ coefficients will in general satisfy. However F' will certainly exist if $(n - k + 1)(\nu - k + 1) \geq n(\nu - \kappa) + 1$ or if

$$n - k \geq (k - 1)(\nu - \kappa). \tag{7}$$

The cases hitherto considered relate to forms F for which $k = \kappa$ and therefore $n = \nu$. Then (7) becomes

$$(2 - k)(n - k) \geq 0. \tag{8}$$

The case $k = \kappa = 1$ presents no interest. But for $k = \kappa = 2$ the relation

(8) is satisfied and the complementary form F' must exist. Thus we find the classic theorem of Poncelet:

(9) *If a form $F_{2,2}$ admits one configuration $\Delta_{n,n}^{2,2}$ it admits infinitely many and is poristic.*

Moreover it is known that poristic forms $F_{2,2}$ with configurations $\Delta_{n,n}^{2,2}$ actually exist for all values $n \geq 3$. Hence the complementary poristic forms also exist, i.e.,

(10) *There exist poristic forms $F_{n\,2,\,n-2}$ with configurations $\Delta_{n,n}^{n-2,\,n=2}$ for all values $n \geq 4$.*

The extreme cases where the orders of F' are as low as possible are worth noting. Thus if $n - k = 0 = \nu - \kappa$ and F' therefore reduces to a constant the relation (7) is satisfied and we can state that

(11) *If a form $F_{k,\kappa}$ admits one configuration $\Delta_{k,\kappa}^{k,\kappa}$ it is completely poristic.*

If $n - k = 1 = \nu - \kappa$ we find from (2) that $k = \kappa$. Then the complementary form F' is $(\beta t)(b\tau)$ and each of the $k+1$ t's determines one of the $k+1$ τ's in the projectivity $F' = 0$. Hence

(12) *If a form $F_{k,k}$ admits one configuration $\Delta_{k+1,\,k+1}^{k,k}$ it will be poristic if and only if furthermore the $k+1$ elements t are projective to the $k+1$ elements τ in the order indicated by the complementary configuration $\Delta_{k+1,\,k+1}^{1,1}$.*

This checks the theorem of Meyer which states that if a tetrahedron is inscribed in a cubic curve C with points t and circumscribed to a cubic curve C' with planes τ the curves will not in general admit ∞^1 such tetrahedra. Meyer's condition for the poristic case is that a second such tetrahedron shall exist. But since the incidence condition of point t and plane τ is $F = (\alpha t)^3(a\tau)^3 = 0$, the theorem (12) furnishes as the simpler additional condition for a porism that the four points of the given tetrahedron shall have the same double ratio on C as the four planes have on C'.

It may be observed that in the poristic case of theorem (12) we may always subject τ to such a linear transformation that the complementary form F' becomes $t - \tau$. Then the form F becomes symmetric and vanishes when t, τ belong to the same set of the involution I_1^{k+1} determined by the pencil, $(\gamma t)^{k+1} + \lambda(\delta t)^{k+1} = 0$. Theorem (12) then reads

(13) *If a symmetric form $F_{k,k}$ admits one configuration $\Delta_{k+1,\,k+1}^{k,\,k}$ it admits ∞^1 such configurations.*

3. *Covariant Porisms.*—In the actual construction of poristic forms certain porisms covariantly connected with the given porism are very useful. They are defined as follows. Let the form $F_{k,\kappa}$ be poristic with configurations $\Delta_{n,\nu}^{k,\kappa}$ whence $FF' = D_{n,\nu}$. Let S, T, U denote respectively the correspondences from t to τ determined by F, F', $D_{n,\nu}$. We use Severi's (Lezioni) definitions for the sum, product, and inverse of these correspondences. Then

$$U = S + T, \qquad U^{-1} = S^{-1} + T^{-1},$$

$$UU^{-1} = (S + T)(S^{-1} + T^{-1}) = SS^{-1} + TT^{-1} + ST^{-1} + TS^{-1}.$$

For these products we may state at once that

(14) *The products* $SS^{-1}, TT^{-1}, UU^{-1}$ *are symmetric correspondences between* t *and* t' *given respectively by forms* $G_{\kappa(k-1,\,\kappa(k-1)}$, $G'_{(\nu-\kappa)(n-k-1),\,(\nu-\kappa)(n-k-1)}$, $I^{\nu}_{n-1,\,n-1}$ *which express that in* $F = 0$, $F' = 0$, $D_{n,\,\nu} = 0$ *respectively* t *and* t' *determine a common* τ. *Here* $I_{n-1,\,n-1} = 0$ *expresses that* t, t' *belong to the same set of the involution* $(\gamma t)^n + \lambda(\delta t)^n = 0$.

(15) *The products* ST^{-1} *and* TS^{-1} *are correspondences between* t, t' *given by forms* $H_{k(\nu-\kappa),\,\kappa(n-k)}$, $H'_{\kappa(n-k),\,k(\nu-\kappa)}$ *which express respectively that* t *in* $F = 0$ *and* t' *in* $F' = 0$, *and* t *in* $F' = 0$ *and* t' *in* $F = 0$, *determine a common* τ. *The forms* H, H' *interchange with* t, t' *since* ST^{-1} *and* TS^{-1} *are inverses.*

Corresponding to the sum of products above we have at once the identity among the forms just defined

$$I^{\nu} = G \cdot G' \cdot H \cdot H'. \tag{16}$$

Evidently if the forms F, F' are poristic and have the closure property this must be true of the covariant forms I, G, G', H, H' as well and these will be called *covariant porisms.* It is clear moreover that all of these forms can be deduced from F by rational processes whence they are *rational covariants* of F.

On interchanging the order of the factors in the above products and interchanging also n, ν; k, κ; t, τ; t', τ' we have correspondences $S^{-1}S$, $T^{-1}T$, $U^{-1}U$, $S^{-1}T$, $T^{-1}S$ determined by forms $\overline{G}$, $\overline{G'}$, $\overline{I}$, $\overline{H}$, $\overline{H'}$ for which

$$\overline{I}^{n} = \overline{G} \cdot \overline{G'} \cdot \overline{H} \cdot \overline{H'}. \tag{17}$$

For a complete porism the correspondence T does not exist and $G = I^{\nu}$ but as a rule it will be necessary that the forms I and $\overline{I}$ factor and that various powers of these factors shall be distributed among the forms G, H in (16) and (17). This is true for example of the Poncelet porisms $\Delta^{2,\,2}_{n,\,n}$ $(n > 3)$ for which I is an $(n-1,\, n-1)$ form and G a $(2, 2)$ form.

An especially symmetric type of porism may occur when $G = I$ and $\overline{G} = \overline{I}$. Then $\kappa(k-1) = n-1$ and $k(\kappa-1) = \nu - 1$. On eliminating n, ν from (2) we find that $(\kappa - k)(\kappa - 1)(k-1) = 0$. Since $\kappa > 1$ and $k > 1$ this requires that $\kappa = k$ and therefore $n = \nu$ and $k(k-1) = n-1$. In this case the configurations are $\Delta^{r+1,\,r+1}_{r^2+r+1,\,r^2+r+1}$'s $(r = k-1)$. If $r = p^j$ where p is a prime a grouping of t's and τ's is immediately suggested by the finite plane geometry (mod p^j) where of the totality of $r^2 + r + 1$ points t and $r^2 + r + 1$ lines τ there are $r + 1$ lines τ on a point t and $r + 1$ points t on a line τ. For such configurations we have the theorem:

(18) *If a form* $F_{r+1,\,r+1}$ $(r \geq 2)$ *admits one configuration* $\Delta^{r+1,\,r+1}_{r^2+r+1,\,r^2+r+1}$ *so arranged that any two* t*'s determine a common* τ *(or vice versa) it admits infinitely many.*

For then the symmetric form G admits a configuration $\Delta^{r^2+r,\,r^2+r}_{r^2+r+1,\,r^2+r+}$ and according to (13) is poristic. Hence F also must have the closure property. This series of cases includes for $r = 2$ that of Professor White.* The question as to whether for values of $r > 2$ F can admit *one* configuration of the sort described in (18) is still open.

4. *The Group Problem of the Porism.*—In order that a form $F_{k,\,\kappa}$ may admit a configuration $\Delta^{k,\,\kappa}_{n,\,\nu}$ it is necessary that the n t's be assorted into ν sets of k each (with which we associate the values τ) in such a way that each t is included in κ of the ν sets. This arrangement into sets must be determined before the conditions of theorem (6) can be applied. The problem of finding such arrangementsbe longs to the theory of finite groups and the solution for given values of n, ν, k, κ is not necessarily unique. Indeed the complications of this tactical problem are the most serious bar to any general discussion of the porisms. Certain series of cases may be treated as a class. Such for example are the Poncelet polygons of n sides, and the checkerboard configurations of n^2 points of Sec. 10. As a rule however each case presents its own peculiarities. Precisely the same tactical problem appears in the formation of irrational (i.e., non-symmetric) invariants of a set of n points in an S_{k-1} of weight ν and degree κ.

If $F_{k,\,\kappa}$ is poristic as above and if we transform its variables by means of the equations

$$t = \frac{(\lambda t')^r}{(\mu t')^r}, \qquad \tau = \frac{(l\tau')^\rho}{(m\tau')^\rho}$$

then the transformed form will likewise have the closure property with configurations $\Delta^{rk,\,\rho\kappa}_{rn,\,\rho\nu}$. The peculiarity of poristic forms which arise by such transformation from simpler forms is that the nr t''s ($\nu\rho$ τ''s) divide into $n(\nu)$ sets of $r(\rho)$ each such that every $t'(\tau')$ in a set determines the same values of $\tau'(t')$. Such a grouping may be described as *imprimitive*. We shall however consider only such configurations for which the elements τ vary with t and vice versa except in the case of the complete porisms.

5. *A Geometric Source of All Poristic Double Forms.*—We shall always choose t so that, in the form $F_{k,\,\kappa}$, $k \leqq \kappa$. We can arrange the terms in $F_{k,\,\kappa}$ according to powers of t as in

$$(19) \qquad t^k(a_0\tau)^\kappa + t^{k-1}(a_1\tau)^\kappa + t^{k-2}(a_2\tau)^\kappa + \cdots + (a_k\tau)^\kappa.$$

Then if t is the parameter of an S_{k-1} u of the rational norm curve N^k in S_k,

* Cf. Coble, *Proceedings of the National Academy of Sciences*, Vol. 2 (1916), p. 530.

$$u_0 = t^k, \qquad u_1 = t^{k-1}, \qquad u_2 = t^{k-2}, \qquad \cdots, \qquad u_k = 1;$$

and if τ is the parameter of a point x on the rational curve R^κ of order κ in S_k,

$$(20) \qquad x_0 = (a_0\tau)^\kappa, \qquad x_1 = (a_1\tau)^\kappa, \qquad x_2 = (a_2\tau)^\kappa, \qquad \cdots, \qquad x_k = (a_k\tau)^\kappa;$$

it is clear that (19), or $F_{k,\kappa} = 0$, furnishes the incidence condition of point x of R^κ with S_{k-1} u of N^k in S_k. Thus every double form may be interpreted as an incidence condition of the dual elements on two rational curves one of which is a rational norm curve.

If $F_{k,\kappa}$ is poristic the values of t are arranged in closed sets of n determined by the involution I_1^n, $(\gamma t)^n + \lambda(\delta t)^n = 0$. In each set of n we take the $\binom{n}{k}$ sets of k values of t. These determine $\binom{n}{k}$ points in S_k, each point being the meet of the k S_{k-1}'s determined on N^k by the k t's. As the set of n t's runs through I_1^n the $\binom{n}{k}$ points run over the *involution curve of* I_1^n *with reference to* N^k. The order of the involution curve is $\binom{n-1}{k-1}$ since the S_{k-1} of N^k determined by $t = t_1$ meets the curve in those points obtained by taking with $t = t_1$ any $k-1$ out of the $n-1$ remaining in the set of I_1^n which contains t_1. If the I_1^n is that associated with a form F, poristic with configurations $\Delta_{n,\nu}^{k,\kappa}$, then, of the $\binom{n}{k}$ points on the involution curve associated with a set of n t's, ν are isolated by the ν values of τ and lie on the rational curve (20) of order κ, whence the involution curve has the rational constituent R^κ. Conversely if the involution curve of I_1^n with reference to N^k has a rational constituent R^κ the incidence condition of point τ of R^κ with S_{k-1} t of N^k is a poristic form. For the n S_{k-1}'s of N^k determined by a set of I^n each meets the involution curve and therefore also its constituent R^κ only in points such that all the S_{k-1}'s of N^k through these points are found in the set of n whence the incidence condition would necessarily have the closure property. Hence

(21) *Every poristic double form is the incidence condition with a rational norm curve of a rational constituent of an involution curve with reference to the norm curve.*

This theorem is particularly useful in determining poristic forms $F_{2,n}$ since the conditions that a plane curve be rational are easy to apply. For the Poncelet polygons of n sides attached to $F_{2,2}$, as is well known, the involution curve is composed of $(n-1)/2$ conics if n is odd and if n is even of $(n-2)/2$ conics and a line.

6. *The Poristic* $F_{2,3}$ *with Configurations* $\Delta_{4,6}^{2,3}$.—We now proceed to prove the existence of a number of poristic forms of orders greater than (2, 2) and in some cases to effect their construction. For forms $F_{2,\kappa}$ with configurations $\Delta_{n,\nu}^{2,\kappa}$ we name the n t's of a configuration $t_0, t_1, \cdots, t_{n-1}$ and

the ν τ's by τ_{ij} if τ_{ij} determines t_i, t_j in $F_{2,\kappa} = 0$. Then for $F_{2,3}$ with a $\Delta^{2,3}_{4,6}$ we have $4t$'s, t_0, t_1, t_2, t_3 and $6\tau_{ij}$'s $(i, j = 0, \cdots, 3;\ i \neq j)$. It is convenient to think of the t's as the sides and the τ's as the vertices of a complete quadrilateral such that in $F_{2,3} = 0$ the sides are paired with vertices on them. In the complementary configuration $\Delta'^{2,3}_{4,6}$ the sides are paired with vertices not on them. The number of pairs in each configuration is 12 and the complementary forms F, F' have 12 coefficients each so that the requirements of theorem (6) can be satisfied by imposing not more than two conditions on the $4t$'s and 6τ's.

To find these conditions we use the covariant correspondences of Sec. 3. Since $SS^{-1} = I$, $TT^{-1} = I$, $ST^{-1} = TS^{-1} = I^2$, the condition (16) is satisfied without imposing any conditions on $\Delta^{2,3}_{4,6}$. For the inverse correspondences we find however that $\bar{I}$ breaks up into two symmetric correspondences $\bar{J}\bar{K}$ such that $\bar{J}$ is satisfied by a pair τ_{ij}, τ_{ik} of adjacent vertices of the quadrilateral and $\bar{K}$ by a pair τ_{ij}, τ_{kl} of opposite vertices. Then $\bar{G} = \bar{G}' = \bar{J}$, $\bar{H} = \bar{H}' = \bar{J}\bar{K}^2$.

Since τ_{ij}, τ_{kl} satisfy the (1, 1) symmetric form $\bar{K}$ they are partners in an involution whose fixed points we shall take to be 0, ∞. Then in non-homogeneous coördinates the six τ's are

$$\begin{matrix} \tau_{01},\ \tau_{02},\ \tau_{03} \\ \tau_{23},\ \tau_{31},\ \tau_{12} \end{matrix} = \begin{matrix} \lambda, & \mu, & \nu \\ -\lambda, & -\mu, & -\nu. \end{matrix}$$

Hence the three τ's which correspond respectively to t_0, t_1, t_2, t_3 are determined by the cubics $(\tau - \lambda)(\tau - \mu)(\tau - \nu)$, $(\tau - \lambda)(\tau + \mu)(\tau + \nu)$, $(\tau + \lambda)(\tau - \mu)(\tau + \nu)$, $(\tau + \lambda)(\tau + \mu)(\tau - \nu)$. Since for all values of t in $F_{2,3}$ we obtain at most three linearly independent cubics, these four must be connected by a linear relation whose coefficients are $(\mu - \nu)(\nu - \lambda)(\lambda - \mu)$, $(\mu - \nu)(\nu + \lambda)(\lambda + \mu)$, $(\mu + \nu)(\nu - \lambda)(\lambda + \mu)$, $(\mu + \nu)(\nu + \lambda)(\lambda - \mu)$. Denote the cubics in τ with these respective coefficients by C_0, C_1, C_2, C_3 whence

$$C_0 + C_1 + C_2 + C_3 \equiv 0.$$

Let us now determine 4 quadratics in t, Q_0, Q_1, Q_2, Q_3 such that

$$Q_0 + Q_1 + Q_2 + Q_3 \equiv 0,$$

and for $t = t_i$, $Q_i = -3\rho$ and $Q_k = \rho$ $(k \neq i)$. Such quadratics are

$$Q_0 = r_1 + r_2 + r_3, \qquad Q_i = -3r_i + r_j + r_k \quad (i, j, k = 1, 2, 3),$$

$$r_i = (t_0 - t_i)(t - t_j)(t - t_k).$$

Hence in

$$C_0Q_0 + C_1Q_1 + C_2Q_2 + C_3Q_3 \tag{22}$$

we have an $F_{2,3}$ which admits the given $\Delta^{2,3}_{4,6}$. Moreover by changing the

sign of τ we get an $F'_{2,3}$ which admits the complementary $\Delta'^{2,3}_{4,6}$. Thus the conditions of theorem (6) are satisfied. The $F_{2,3}$ (22) and one of its configurations have the 9 constants, $t_0, \cdots, t_3, \lambda, \mu, \nu, 0, \infty$ whence F alone has 8. Therefore

(23) *The form $F_{2,3}$ in* (22), *subject to three conditions and containing* 8 *constants or two absolute constants is poristic and admits ∞^1 configurations $\Delta^{2,3}_{4,6}$. It is the only type which admits such configurations.*

The degenerate configurations are worth noting. Let as in Sec. 5 t be a line of the norm conic K and τ a point of the rational involution cubic curve C. Suppose that on the conic t_0 and t_1 coincide. Then τ_{01} is a common point of K and C. Moreover now the six τ's are $\left(\begin{smallmatrix} a & -b & -b \\ -a & b & b \end{smallmatrix}\right)$. Thus t_2, t_3 are lines of C since on them two intersections coincide. The pencil of sextics in τ is a pencil of cubics in τ^2 so that it happens four times that $\tau^2 = b^2$ counts twice. Hence the four configurations of the above type account for four of the six common points and the eight common lines of C and K.

In the pencil of sextics there will be either one other sextic with two double roots at 0, ∞ or two sextics each with one double root at 0, ∞ respectively. In the pencil of quartics t there are six which have a double root and four of these have been accounted for. If another has only one double root it leads as above to further common lines of C and K. But these have all been accounted for. Hence there is one further quartic with two double roots $t_0 = t_1$, $t_2 = t_3$ and the six τ's are $\left(\begin{smallmatrix} \tau_{01} & 0 & \infty \\ \tau_{23} & 0 & \infty \end{smallmatrix}\right)$. Hence the two tangents t_0, t_2 of K meet in the node 0, ∞ of C and the points τ_{01}, τ_{23} of C are the two further common points of C and K. Thus C is merely the involution curve of an I_1^4 on a conic K which has acquired a node due to the fact that the I_1^4 contains a perfect square.

7. *The Poristic $F_{2,4}$ with Configurations $\Delta^{2,4}_{5,10}$.*—The case treated in the preceding section can be extended one step as indicated in the closing sentence. The involution curve of an I_1^5 on a conic K is a Lüroth quartic curve. If the involution contains three members each with two double roots then the quartic involution curve is rational. If τ_{ij} is a parameter on this rational curve C such that the tangents t_i, t_j of K meet in the point τ_{ij} of C then there exist ∞^1 configurations $\Delta^{2,4}_{5,10}$ consisting of 5 tangents of K whose 10 meets are on C, and the incidence condition of tangent t_i of K and point τ_{ij} of C, an $F_{2,4}$ form, will admit these configurations. The general I_1^5 with 8 constants is here subject to three conditions to account for the three pairs of double roots so that the poristic $F_{2,4}$ has five constants or two absolute constants.

If in a set of I_1^5, $t_0 = t_1$, $t_2 = t_3$ then the tangent t_4 of K is a double tangent of C while the contacts of tangents t_0, t_2 of K with K are points of C.

If in a set of I_1^5, $t_0 = t_1$ then the lines t_2, t_3, t_4 are common tangents of C and K and the point t_0 of K is a common point of C and K. The first case happens three times, the second case twice, so that the common points and lines of C and K are accounted for.

The existence of this case implies the existence of the complementary poristic form $F_{3,6}$ with configurations $\Delta_{5,10}^{3,6}$.

The I_1^3 (with Poncelet conics K, C), the I_1^4, and the I_1^5 are the only involutions with a sufficient number of double elements to allow the *complete* involution curve C to acquire a sufficient number of double or multiple points not on the conic K to become rational. Perhaps the series may be continued by allowing multiple points of C to appear on K. We shall however pass on to other configurations for which the involution curve C has a rational part.

8. *The Poristic $F_{2,3}$ with Configurations $\Delta_{6,9}^{2,3}$.*—There are but two distinct ways in which six t's, $t_1, \cdots, t_6$ can be arranged in nine pairs in such a manner that each t occurs in three pairs. For both ways the t's divide into two sets of three arranged as in a matrix

$$\begin{pmatrix} t_1 & t_2 & t_3 \\ t_4 & t_5 & t_6 \end{pmatrix}.$$

In the one way each t is paired with a t in the same row or column to form a pair τ_{ij}. For such a configuration the form $F_{2,3}$ could not be poristic. In fact the symmetric (3, 3) form G of SS^{-1} which coördinates to the line t_1 of K the lines t_4, t_2, t_3 would break up into a (2,2) form for which t_1 determines t_2, t_3 and a (1, 1) form for which t_1 determines t_4. Thus the rational cubic C determined by G would factor into a conic and a line which would require that the form $F_{2,3}$ should factor.

The second grouping which, as we shall see, leads to a poristic $F_{2,3}$ coördinates a t in one row to each of the t's in the other row, i.e., t_i ($i = 1, 2, 3$) and t_j ($j = 4, 5, 6$) determine τ_{ij}. If t is a line of a conic K and τ_{ij} a point of the rational cubic C then the 9-point configuration τ_{ij} is cut out on C by two triads of tangents of K. There are on C ∞^4 such 9-point configuration, but of these only ∞^3 for which the six lines touch a conic. The form $F_{2,3}$ if poristic will admit ∞^1 of these configurations. Let us study under this hypothesis the necessary behavior of the covariant porisms. The form G symmetric of degree three in t, t' attached to SS^{-1} will for $t = t_1$, or t_2, or t_3 determine the triad $t' = t_4, t_5, t_6$; and for $t = t_4$, or t_5, or t_6 the triad $t' = t_1, t_2, t_3$. Hence as the $\Delta_{6,9}^{2,3}$'s more around we have a pencil of cubics $(\gamma t)^3 + \lambda(\delta t)^3$ such that this member of the pencil determines involutorily another member $(\gamma t)^3 - \lambda(\delta t)^3$. This can be visualized by taking sections of a space cubic curve by planes on a line where the planes π_1, π_2 cut out the

fixed members $(\gamma t)^3$, $(\delta t)^3$ of the above involution. Then points t, t' on the space cubic are apolar to the pair of planes π_1, π_2 if

$$G = (\gamma t)^3(\delta t')^3 + (\gamma t')^3(\delta t)^3 = 0.$$

Interpreting this form, which is evidently poristic, on K, the point t, t' runs over a cubic C of genus one. The form G has three absolute constants which arise from π_2 after the space cubic and π_1 have been chosen. Hence the cubic C and conic K together have eleven constants, or given C there are ∞^2 conics K each with $\infty^1 \Delta^{2,3}_{6,9}$'s. Thus the ∞^3 configurations whose six lines touch a conic accounted for and it is clear that the existence of one such configuration of six t's for a symmetric (3, 3) form implies the existence of infinitely many.

The cubic C must acquire a node in order that its points may be named by a parameter τ_{ij}. This requires that in one set of six t's, $t_1 = t_2$ and $t_4 = t_5$, or that π_1, π_2 be a pair of planes apolar to two tangent planes of the space cubic on the common line of π_1, π_2, which is one condition. Hence

(24) *There exist poristic forms $F_{2,3}$ with configurations $\Delta^{2,3}_{6,9}$ depending upon two absolute constants. The existence of one such configuration implies the porism.*

For given F and one configuration Δ the equation of the rational cubic C referred to K is a symmetric (3, 3) form G which admits one $\Delta^{3,3}_{3,3}$ and therefore according to (11) is poristic. This requires in turn that F be poristic.

In the pencil of cubics $(\gamma t)^3 + \lambda(\delta t)^3$ there are four members with a double root but two of these are paired at the double point so that we have one case

$$\begin{pmatrix} t_1 & t_1 & t_3 \\ t_4 & t_4 & t_6 \end{pmatrix}.$$

The double point is either $\tau_{14} = \tau_{25}$ or $\tau_{15} = \tau_{24}$ with further coincidences $\tau_{16} = \tau_{26}$ and $\tau_{34} = \tau_{35}$ so that the lines t_3, t_6 are common tangents of C and K. There remain two cases

$$\begin{pmatrix} t_1 & t_1 & t_3 \\ t_4 & t_5 & t_6 \end{pmatrix}.$$

Then $\tau_{14} = \tau_{24}$, $\tau_{15} = \tau_{25}$, $\tau_{16} = \tau_{26}$ and the lines t_4, t_5, t_6 account for the remaining common tangents of C and K. Moreover corresponding to the double cubics of the involution, $(\gamma t)^3$ and $(\delta t)^3$, there are two cases

$$\begin{pmatrix} t_1 & t_2 & t_3 \\ t_1 & t_2 & t_3 \end{pmatrix}.$$

Then $\tau_{15} = \tau_{24}$, $\tau_{16} = \tau_{34}$, $\tau_{26} = \tau_{35}$ and the points t_1, t_2, t_3 of K are on C. Thus the common points of C and K are located. Moreover the pencil of

9-ics τ will have sixteen members with double roots and all of these appear above.

The configuration $\Delta^{2,3}_{6,9}$ is a special case of the *checkerboard configuration* of n^2 points which are the intersections of one set of n lines with another set of n lines. These are discussed further in the next two sections.

9. *The Poristic $F_{2,4}$ with Checkerboard Configurations $\Delta^{2}_{8},{}^{4}_{16}$.*—If a quartic Q has an inscribed checkerboard cut out by the two sets of four lines Q_1, Q_2 then Q has the form $Q_1 + \mu Q_2 = 0$. This form contains $2 \times 8 + 1$ or 17 constants whereas Q has but 14 so that a given quartic has presumably ∞^3 checkerboards. We should expect therefore a finite number for which the eight lines touch a conic K. But as a matter of fact if one set of eight lines touches K there will be ∞^1 sets which touch K. For if $(\gamma t)^4$ and $(\delta t)^4$ represent these sets of four lines on K, the equation of Q in Darboux coördinates t, t' is $(\gamma t)^4(\delta t')^4 + (\gamma t')^4(\delta t)^4 = 0$ and this form is poristic. This pair of binary quartics on K has five absolute constants whence Q is subject to one invariant condition. Thus we find the theorem analogous to the Lüroth theorem on the inscribed pentagons of a quartic curve:

(25) *If a quartic Q passes through the sixteen points of one checkerboard whose eight lines touch a conic K then Q is subject to one invariant condition and has ∞^1 inscribed checkerboards whose lines envelop K.*

The two sets of four lines which make up the checkerboard are pairs of sets of an I_1^4 on K which are involutorily related. If Q should have three double points and be rational with parameter τ then the condition that point τ of Q and line t of K be incident is a poristic $F_{2,4}$. If a binary quartic of the pencil has a double root while the paired quartic has not, i.e., if the pair of quartics is $\left(\begin{smallmatrix} a & a & b & c \\ \alpha & \beta & \gamma & \delta \end{smallmatrix}\right)$ then $\alpha, \beta, \gamma, \delta$ are common tangents of K and Q. If both quartics of the pair have a double root as in $\left(\begin{smallmatrix} a & a & b & c \\ a & a & \beta & \gamma \end{smallmatrix}\right)$ then the point (a, α) is a double point of Q and the lines b, c, β, γ are common tangents. If both quartics of the pair have a triple root as in $\left(\begin{smallmatrix} a & a & a & b \\ a & a & a & \beta \end{smallmatrix}\right)$ then the point (a, α) is a triple point of Q while b, β are common tangents of K and Q which are inflexional for the latter.

If then the case $\left(\begin{smallmatrix} a & a & b & c \\ a & a & \beta & \gamma \end{smallmatrix}\right)$ occurs three times Q has three nodes and twelve tangents in common with K. All the multiple roots of the pencil are accounted for but clearly the six quartics of the pencil with double roots must have their parameters in involution in order that the fixed pair $(\gamma t)^4$, $(\delta t)^4$ may exist. Such a pencil has only two absolute constants.

If the case $\left(\begin{smallmatrix} a & a & a & b \\ a & a & a & \beta \end{smallmatrix}\right)$ occurs there will remain two cases $\left(\begin{smallmatrix} a & a & b & c \\ \alpha & \beta & \gamma & \delta \end{smallmatrix}\right)$ so that Q has a triple point and $2.4 + 2.2$ tangents in common with K. Then $(\gamma t)^4$, $(\delta t)^4$ is any one of the ∞^1 pairs of quartics whose parameters are harmonic to the two with triple points so that for this case also there are two absolute constants.

(26) *There are two types, each with two absolute constants, of poristic* $F_{2,4}$*'s with configurations* $\Delta_{8,16}^{2,4}$ *of the checkerboard type. The checkerboards are inscribed in a quartic which for the one type has three nodes; for the other, a triple point.*

10. *The Poristic* $F_{2,n}$ *with Checkerboard Configurations* $\Delta_{2n,n^2}^{2,n}$.—Let us generalize the configuration of the preceding section to $2n$ lines and n^2 points and inquire as to the existence of porisms with these configurations. As before they must be associated with two particular members of a pencil of binary n-ics if the $2n$ lines envelop a conic K. An n-ic curve and one inscribed checkerboard has $4n + 1$ constants. It is $2n - 5$ conditions that the $2n$ lines envelop K so that then the curve and configuration have $2n + 6$ constants or $2(n - 1)$ absolute constants. However it is then determined by two fixed members of a pencil of binary n-ics which have $2(n - 1) - 1$ absolute constants. Hence

(27) *If a curve* C *of order* n *has an inscribed checkerboard of* n^2 *points and* $2n$ *lines which envelop a conic* K *then there are* ∞^1 *such checkerboards inscribed in* C *and circumscribed to* K. *The curve* C *has* $2n - 3$ *absolute constants.*

Let us now attempt to make C rational by giving it nodes. Suppose that one member of the pencil of n-ics has k double roots and that the paired members has l double roots. Then each double root of the one member with each double root of the other member determines a node of C so that this pair of n-ics contributes $k + l$ of the $2(n - 1)$ double roots of the pencil and contributes kl nodes to C. If C is rational for r such cases we have

$$\begin{aligned} k_1 + l_1 + k_2 + l_2 + \cdots + k_r + l_r &= 2(n-1), \\ k_1l_1 + k_2l_2 + \cdots + k_rl_r &= (n-1)(n-2)/2. \end{aligned} \tag{28}$$

In each pair of this sort the $n - 2k$ simple roots of the first member will each contribute an l-fold common tangent of C and K. Since it is a consequence of (28) that

$$(n - 2k_1)l_1 + (n - 2l_1)k_1 + \cdots + (n - 2k_r)l_r + (n - 2l_r)k_r = 4(n - 1),$$

all the common tangents of C and K are found.

It is however $k - 1$ conditions on the pencil that a member shall have k double roots, $l - 1$ conditions that another shall have l double roots, and one condition that these be paired. Hence the number of absolute constants which remain is

$$2n - 3 - \sum_{i=1}^{r} (k_i + l_i - 1) = 2n - 3 - \{2(n-1) - r\} = r - 1.$$

An evident inequality for the integers k, l is

$$0 \leqq k_i \leqq n/2, \qquad 0 \leqq l_i \leqq n/2. \tag{29}$$

One solution of the diophantine system (28), (29) is

$$(30)\quad n = 2j+1\begin{cases} k_1, l_1 = j, j \\ k_2, l_2 = j, j-1; \\ k_3, l_3 = 1, 0 \end{cases} \qquad n = 2j\begin{cases} k_1, l_1 = j, j-1 \\ k_2, l_2 = j-1, j-1 \\ k_3, l_3 = 1, 0. \end{cases}$$

Here $r - 1 = 2$ so that these forms have two absolute constants. But there may well be other solutions of the system.

If the diophantine system above be extended to take account of the possibility of points of multiplicity as great as $n - 1$ on C, the system becomes much more complicated but solutions of this higher type exist. For example construct the pencil of n-ics by taking an n-ic with $(n - 1)$-fold root a and simple root b and another with $(n - 1)$-fold root α and simple root β (one absolute constant) and choose for $(\gamma t)^n$, $(\delta t)^n$ a pair of n-ics of the pencil whose parameters are apolar to the two with $(n - 1)$-fold roots (a second absolute constant). Then C has an $(n - 1)$-fold point at (a, α) and the tangents b, β of K have $(n - 1)$-point contact with C. Hence we have the generalized form of (26).

(31) *There will exist poristic forms $F_{2,n}$ with configurations $\Delta^{2,n}_{2n,n^2}$ of the checkerboard type whose $2n$ lines t envelop a conic K while the n^2 points τ describe a rational n-ic curve C whose multiple points may consist only of double points as in* (30), *or of a single $(n - 1)$-fold point, or of points with intermediate multiplicities. The extreme cases actually constructed above both involve two absolute constants.*

11. *The Poristic $F_{2,n-1}$ with Configurations $\Delta^{2,n-1}_{2n,n(n-1)}$.*—If in the checkerboard of $2n$ lines and n^2 points a diagonal be selected this implies an involutory mutual ordering of the two sets of n lines. Thus if we name the lines $\binom{t_1 t_2 \cdots t_n}{s_1 s_2 \cdots s_n}$ and the n^2 points τ_{ij} $(i, j = 1, \cdots, n)$ we may select as diagonal points the n points τ_{ii}. If this diagonal be deleted the resulting configuration will have $2n$ lines as before but only $n^2 - n$ points τ_{ij} $(i \neq j)$. If ∞^1 such configurations were circumscribed to K there would necessarily exist on K an involution I which may be taken as $t' = -t$. Then the form $(\delta t)^n$ of the preceding section would arise from $(\gamma t)^n$ by changing the sign of t. The set of $2n$ lines would have parameters $\binom{t_1\ t_2 \cdots\ t_n}{-t_1 -t_2 \cdots -t_n}$. From the curve C there would factor the line joining the fixed points 0, ∞ of the involution I on K. Of the various rational curves C referred to in (31) let us consider only the one with an $(n - 1)$-fold point determined by $\binom{a,\ a, \cdots, a,\ b}{-a, -a, \cdots, -a, -b}$ for which $(\gamma t)^n = (t - a)^{n-1}(t - b)$, $(\delta t)^n = (t + a)^n(t + b)$. After factoring out the line of I there remains a curve C' of order $n - 1$ with an $(n - 2)$-fold point $(a, -a)$. In the equation of C' we have the one absolute constant implied in 0, ∞, a, b. If τ_{ij} is a parameter on the rational curve C' we find from the incidence condition of C' and K that

(32) *There exist poristic forms* $F_{2,n-1}$, *involving at least one absolute constant, with configurations* $\Delta^{2,n-1}_{2n,n(n-1)}$ *of the type* t_i, s_j *and* τ_{ij} *determined by* t_i, s_j $(i, j = 1, \cdots, n;\ i \neq j)$.

12. *The Poristic* $F_{3,4}$ *with Box Configurations* $\Delta^{3,4}_{6,8}$.—As an instance of poristic double forms with distinct orders both greater than two we take the $F_{3,4}$. Unless this is completely poristic with configurations $\Delta^{3,4}_{3,4}$, the simplest configurations it can admit are $\Delta^{3,4}_{6,8}$'s. For these a grouping immediately at hand is the six faces t and eight vertices τ of a cube or box where $F_{3,4} = 0$ coördinates to each vertex the three faces on it and to each face the four vertices on it. If we call the three pairs of opposite faces $t_1, t_2;\ t_3, t_4;\ t_5, t_6$, the vertices may be named $\tau_{135}, \tau_{246};\ \tau_{136}, \tau_{245};\ \tau_{145}, \tau_{236};\ \tau_{146}, \tau_{235}$ where τ_{ijk} is the vertex on faces t_i, t_j, t_k.

If $F_{3,4}$ is poristic the correspondence SS^{-1} associates $t_3^2, t_4^2, t_5^2, t_6^2$ to t_1. Hence the symmetric form G is a perfect square and $\sqrt{G}$ is a rational symmetric (4, 4) correspondence. In UU^{-1} t_1 corresponds to t_2, t_3, t_4, t_5, t_6 whence the form I factors into $J\sqrt{G}$ where J is a symmetric (1, 1) correspondence or involution whose pairs are opposite faces. Let this involution be $t' = -t$ so that the six faces may be taken as $\pm l, \pm m, \pm n$.

The correspondence $S^{-1}S$ makes τ_{135} correspond to $\tau^2_{136}\tau^2_{145}\tau^2_{235}\tau_{146}\tau_{236}\tau_{245}$ whence the form G factors into K^2L where K, L are symmetric (3, 3) forms which coördinate vertices joined by an edge or by a face diagonal respectively. Moreover $U^{-1}U$ makes τ_{135} correspond to $\tau_{246};\ \tau_{136}, \tau_{145}, \tau_{235};\ \tau_{146}, \tau_{236}, \tau_{245}$ or $\bar{I} = MKL$ where $M = 0$ is the involutory correspondence between opposite vertices. Let us take M to be $\tau' = -\tau$ whence the eight τ's may be taken as $\pm\alpha, \pm\beta, \pm\gamma, \pm\delta$ and the form F must be unaltered by the simultaneous change of sign of t and τ.

In the configuration thus simplified the form F must coördinate the faces and vertices as follows:

$$l\colon -\alpha,\ \beta, -\gamma,\ \delta;\quad m\colon\ \alpha,\ \beta, -\gamma, -\delta;\quad n\colon -\alpha,\ \beta,\ \gamma, -\delta$$

$$-l\colon\ \alpha, -\beta,\ \gamma, -\delta;\quad -m\colon -\alpha, -\beta,\ \gamma,\ \delta;\quad -n\colon\ \alpha, -\beta, -\gamma,\ \delta.$$

In order to be unaltered by changing the signs of t, τ and to have the solutions indicated for $t = \pm l, \pm m$, $F_{3,4}$ must be

$$\begin{aligned} r\{&(\tau+\alpha)(\tau-\beta)(\tau+\gamma)(\tau-\delta)/(t-l) \\ &\quad -(\tau-\alpha)(\tau+\beta)(\tau-\gamma)(\tau+d)/(t+l)\} \\ +\, s\{&(\tau-\alpha)(\tau-\beta)(\tau+\gamma)(\tau+\delta)/(t-m) \\ &\quad -(\tau+\alpha)(\tau+\beta)(\tau-\gamma)(\tau-\delta)/(t+m)\} = 0. \end{aligned} \tag{33}$$

In addition for $t = n$ $F_{3,4}$ must vanish for $\tau = -\alpha, \beta, \gamma, -\delta$ and then

will have the proper solutions for $t = -n$. These conditions lead to the equations

$$\frac{r}{s} = \frac{(n+l)}{(n-m)}\frac{(\beta+\alpha)}{(\beta-\alpha)}\frac{(\gamma-\alpha)}{(\gamma+\alpha)} = -\frac{(n+l)}{(n+m)}\frac{(\beta+\alpha)}{(\beta-\alpha)}\frac{(\beta-\delta)}{(\beta+\delta)}$$

$$= -\frac{(n-l)}{(n-m)}\frac{(\gamma-\alpha)}{(\gamma+\alpha)}\frac{(\gamma+\delta)}{(\gamma-\delta)} = \frac{(n-l)}{(n+m)}\frac{(\beta-\delta)}{(\beta+\delta)}\frac{(\gamma+\delta)}{(\gamma-\delta)},$$

which are satisfied by

$$(34)\qquad \begin{gathered} l : m : n = \frac{1}{(\gamma\alpha-\beta\delta)} : \frac{1}{(\gamma\delta-\alpha\beta)} : \frac{1}{(\alpha\delta-\beta\gamma)}; \\ \frac{r}{s} = \frac{(\alpha+\beta)}{(\alpha+\gamma)}\frac{(\gamma+\delta)}{(\beta+\delta)}\frac{(\alpha\beta-\gamma\delta)}{(\alpha\gamma-\beta\delta)}. \end{gathered}$$

Hence the form $F_{3,4}$ determined in (33), (34) admits one $\Delta^{3,4}_{6,8}$. Moreover the form $F'_{3,4}$ which arises from $F_{3,4}$ by the change of sign of t alone admits the complementary configuration whence according to (6) the form F is poristic. Two constants for t, τ each have been absorbed in the choice of the involutions $t' = -t$, and $\tau' = -\tau$. By proper choice of the unit points two of the four constants $\alpha, \beta, \gamma, \delta$ can be absorbed leaving two absolute constants for the form $F_{3,4}$ and one configuration, or one absolute constant for the form.

(35) *There exists a poristic form $F_{3,4}$ involving one absolute constant with configurations $\Delta^{3,4}_{6,8}$ of the box type; or there is in space a system of ∞^1 boxes whose faces envelop a space cubic curve and whose corners run over a rational quartic curve.*

With regard to this space quartic curve we remark that its sections and therefore their common apolar quartic all admit the involution $\tau' = -\tau$ and are therefore harmonic. The quartic therefore has a node and no absolute constant. Hence the constant in $F_{3,4}$ must be due to the choice of the series of boxes inscribed in the quartic and each of the ∞^1 series determines its own cubic curve. This situation perhaps deserves further study.

It is apparent that the box is the simplest instance in space of a "cellular configuration" (with one cell), the extension to space of the checkerboard configuration in the plane. It would be interesting to know whether forms F_{3,n^2} with cellular configurations Δ^{3,n^2}_{3n,n^3} containing $(n-1)^3$ cells occur. We shall not however attempt to multiply further examples of poristic forms but will close with a summary of the poristic cases which have been established thus far.

13. *Summary of Poristic Double Forms.*

	Configuration	*Group type*	*Reference*
1°	$\Delta^{2,\,2}_{n,\,n}$	Poncelet polygon	(9)
2°	$\Delta^{n-2,\,n-2}_{n,\,n}$	Complement of 1°	(10)
3°	$\Delta^{k,\,\kappa}_{k,\,\kappa}$	Complete porism	(11)
4°	$\Delta^{k,\,k}_{k+1,\,k+1}$	Transform of I_1^{k+1}	(12,) (13)
5°	$\Delta^{3,\,3}_{7,\,7}$	White's porism	(18)
6°	$\Delta^{4,\,4}_{7,\,7}$	Complement of 5°	
7°	$\Delta^{2,\,3}_{4,\,6}$	Quadrilateral porism	(23)
8°	$\Delta^{2,\,4}_{5,\,10}$	Pentagonal porism	Sec. 7
9°	$\Delta^{3,\,6}_{5,\,10}$	Complement of 8°	
10°	$\Delta^{2,\,n}_{2n,\,n^2}$	Checkerboard	(24), (26), (31)
11°	$\Delta^{2,\,n-1}_{2n,\,n(n-1)}$	Checkerboard without diagonal	(32)
12°	$\Delta^{2(n-1),\,n(n-1)}_{2n,\,n^2}$	Complement of 10°	
13°	$\Delta^{2(n-1),\,(n-1)^2}_{2n,\,n(n-1)}$	Complement of 11°	
14°	$\Delta^{3,\,4}_{6,\,8}$	Box porism	(35)

14. *Extensions to Multiple Forms.*—The double binary forms may be extended in various ways. One natural extension of the (2, 2) incidence condition of line t and point τ of two conics is to the incidence condition of plane and point of two quadrics. We should then inquire as to the existence of polyhedra circumscribed to the one quadric and inscribed in the other. However on naming the planes of one quadric by the parameters (of its generators) λ, μ, and the points of the other by λ', μ' the incidence condition appears in the form

$$(a\lambda)(b\mu)(c\lambda')(d\mu') = 0$$

in which unfortunately the four variables are not on a par.

More interesting, at least from the algebraic point of view, are the triple binary forms,

$$(a\lambda)^l(b\mu)^m(c\nu)^n = 0. \tag{36}$$

Let λ_0, μ_0, ν_0 be any solution of (36). Then given λ_0, μ_0 we obtain $n-1$ values of ν in addition to ν_0 which satisfy (36); and from any one of these n solutions new solutions with different λ or with different μ are obtained. If after a time no *new* solutions can be obtained by this process from those already known then we shall have a triple form with the *closure* property.

Two types of triple forms with the closure property are suggested by earlier cases. The first is an extension of the form G of Sec. 8. Let the parameters λ, λ', λ'' which determine n-ics in the pencils

$$(\gamma t)^n + \lambda(\delta t)^n = 0, \qquad (\gamma t')^n + \lambda'(\delta t')^n = 0, \qquad (\gamma t'')^n + \lambda''(\delta t'')^n = 0$$

be connected by the involutory relation

$$a_0\lambda\lambda'\lambda'' + a_1(\lambda'\lambda'' + \lambda''\lambda + \lambda\lambda') + a_2(\lambda + \lambda' + \lambda'') + a_3 = 0.$$

On replacing the $\lambda, \lambda', \lambda''$ by their values in terms of t, t', t'' we obtain a triple (n, n, n) form with the closure property.

Another triple form with the closure property is the determinant

$$\Delta = \begin{vmatrix} (a_0\lambda)^l & (a_1\lambda)^l & (a_2\lambda)^l \\ (b_0\mu)^m & (b_1\mu)^m & (b_2\mu)^m \\ (c_0\nu)^n & (c_1\nu)^n & (c_2\nu)^n \end{vmatrix} = 0 \tag{37}$$

obtained by eliminating the line coördinates ξ_0, ξ_1, ξ_2 in S_2 from the equations

$$\begin{aligned} (a_0\lambda)^l\xi_0 + (a_1\lambda)^l\xi_1 + (a_2\lambda)^l\xi_2 &= 0, \\ (b_0\mu)^m\xi_0 + (b_1\mu)^m\xi_1 + (b_2\mu)^m\xi_2 &= 0, \\ (c_0\nu)^n\xi_0 + (c_1\nu)^n\xi_1 + (c_2\nu)^n\xi_2 &= 0, \end{aligned} \tag{38}$$

which represent rational curves in S_2 of orders l, m, n and parameters λ, μ, ν respectively. The condition $\Delta = 0$ expresses that points λ, μ, ν respectively on these curves are collinear and obviously Δ has the closure property.

If Δ factors then each factor will likewise have the closure property. Special cases of such factorization are furnished by the involution determined by the rational plane curve of order k, $x_i = (d_it)^k$ $(i = 0, 1, 2)$. The condition that points t_0, t_1, t_2 of this curve lie on a line is the vanishing of the determinant

$$|(d_it_j)^k| = (t_0t_1)(t_0t_2)(t_1t_2)\cdot(\alpha_0t_0)^{k-2}(\alpha_1t_1)^{k-2}(\alpha_2t_2)^{k-2}.$$

Here the symmetric $(k-2, k-2, k-2)$ form has the closure property. Similarly if in Δ, $(b_i\mu)^m_{\mu=\nu} = (c_i\nu)^n$, then the factor $(\mu\nu)$ appears and the remaining factor is an $(l, m-1, m-1)$ form symmetric in μ, ν with the closure property. It is clear however from a discussion of these special cases* that the existence of one closed set is not in general sufficient to ensure closure.

If in the form (36) λ, μ, ν be regarded as parameters in three plane pencils on lines L, M, N respectively in S_3, and λ, μ, ν as the coördinates of the intersection of the three planes, then (36) is the equation of a surface of order $l+m+n$ with the lines L, M, N as l-, m-, n-fold lines respectively. If the form has the closure property then the points of the surface divide into closed sets such that the bisecants from any point of the set across any two of the lines L, M, N meet the surface again in points of the set. A striking case of this sort is that of the (2, 2, 2) form for which the bisecant construction for lines M, N leads to an involutory birational transformation I_L of the sextic surface into itself. The closure case occurs when the three involutions I_L, I_M, I_N generate a finite group.

* Cf. Coble, "Symmetric Binary Forms and Involutions," this JOURNAL, Vol. 32 (1910), p. 333; in particular (104) p. 336.

Three instances of such sextic surfaces may be given. If first in (37) $l = m = n = 2$ the involutions I_L, I_M, I_N generate an abelian G_8. The closed sets of eight points are cut out by three pairs of planes on L, M, N. If second in (37) $l = 2$, $m = n = 3$, and $(b_i\mu)^3_{\mu=\nu} = (c_i\nu)^3$ then the quadric $(\mu\nu)$ factors out leaving a sextic surface with closed sets of 12 points. Now I_μ and I_ν generate a dihedral G_6 and I_L, interchangeable with I_M and I_N generates with G_6 a dihedral G_{12}. If thirdly in (37) $l = m = n = 4$, $(c_i\nu)^4_{\nu=\lambda} = (a_i\lambda)^4$ and, $(b_i\mu)_{\mu=\lambda} = (a_i\lambda)^4$ then the sextic $(\mu\nu)(\nu\lambda)(\lambda\mu)$ factors out leaving a sextic surface with closed sets of 24 points. The involutions now generate a G_{24} isomorphic with the symmetric G_{24}.

These fairly evident cases where factorization of Δ is possible can be generalized to higher orders or to larger numbers of variables. Though I have not been able to find other types of factorization it would seem to be quite likely that such new types exist and their determination presents a fascinating problem.

URBANA, ILL.,
May, 1920.

EINSTEIN'S THEORY OF GRAVITATION: DETERMINATION OF THE FIELD BY LIGHT SIGNALS.

By Edward Kasner.

In this first paper we discuss the determination of a four-dimensional manifold

$$ds^2 = \Sigma g_{ik} dx_i dx_k \qquad (i, k = 1, 2, 3, 4) \tag{1}$$

obeying Einstein's equations of gravitation $G_{\mu\nu} = 0$, when we are given merely *the light equation*

$$\Sigma g_{ik} dx_i dx_k = 0.$$

This means that the ratios of the ten potentials g_{ik} are given and the problem is to determine the potentials themselves so that the gravitational equations are fulfilled.

The result shows that for example the solar gravitational field, or any field assumed to differ only slightly from the galilean field or flat space, can be completely explored by light signals alone. The corresponding determination by orbits alone will be given later. *The light rays determine the orbits, and vice versa.* A statement of results for both problems was given in *Science*, October 29, 1920 (vol. 52, pp. 413–14), in particular the connection between the planetary and light observations for the solar field.

We have also shown that the (exact) solar field can be regarded as immersed in a flat space of 6 dimensions; but that no solution of the Einstein equations can be obtained from flat space of 5 dimensions. (Cf. abstracts in *Bull. Amer. Math. Soc.*, vol. 27, pp. 62 and 102.)

§ 1. General Equations.

In order to write the general equations of gravitation we need the following symbolism.

Let g denote the determinant of fourth order formed from the coefficients $g_{\mu\nu}$; and let $g^{\mu\nu}$ denote the minor of the element $g_{\nu\mu}$ divided by g.

The Christoffel three-index symbol of the first kind is

$$[\alpha\beta, \gamma] = \frac{1}{2}\left(\frac{\partial g_{\alpha\gamma}}{\partial x_\beta} + \frac{\partial g_{\beta\gamma}}{\partial x_\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x_\gamma}\right); \tag{2}$$

and that of the second kind is

$$\{\alpha, \beta, \gamma\} = g^{\gamma\epsilon}[\alpha\beta, \epsilon], \tag{3}$$

where, by the regular convention of the Einstein literature, summation is to be understood, the repeated index ϵ stating the values 1, 2, 3, 4.

The Riemann-Christoffel curvature tensor is

$$(4) \quad B^{\rho}_{\mu\nu\sigma} = \{\mu\sigma, \epsilon\}\{\epsilon\nu, \rho\} - \{\mu\nu, \epsilon\}\{\epsilon\sigma, \rho\} + \frac{\partial}{\partial x_\nu}\{\mu, \sigma\rho\} - \frac{\partial}{\partial x_\sigma}\{\mu\nu, \rho\},$$

which is also denoted by the four-index symbol $\{\mu\rho, \sigma\nu\}$.

The "contracted" Riemann-Christoffel tensor (which might well be termed the Einstein tensor) is then

$$(5) \quad G_{\mu\nu} = B^{\rho}_{\mu\nu\rho},$$

summation understood with respect to ρ.

This can be reduced to

$$(6) \quad G_{\mu\nu} = \{\mu\alpha, \beta\}\{\nu\beta, \alpha\} - \frac{\partial}{\partial x_\alpha}\{\mu\nu, \alpha\} + \frac{\partial^2 L}{\partial x_\mu \partial x_\nu} - \{\mu\nu, \alpha\}\frac{\partial L}{\partial x_\alpha},$$

where summations are taken with respect to α and β, and where

$$L = \log \sqrt{-g}.$$

In the actual world with real coördinates, the determinant g is negative since the fundamental quadratic form has three negative dimensions and one positive dimension (as in the Lorentz-Minkowski world based on the affine-euclidean form

$$dt^2 - dx^2 - dy^2 - dz^2,$$

that is, the special relativity theory). The results which follow are not dependent on this reality assumption. We may write in general

$$L = \tfrac{1}{2}\log g,$$

since only the derivatives of L are involved.

Einstein's law of gravitation (in empty space, the only case we shall here consider) is expressed by the set of ten equations

$$G_{\mu\nu} = 0,$$

involving the second derivatives of the ten functions g_{ik}. [This set takes the place of the single equation

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} = 0,$$

which is the basis of the Newtonian theory.]

Any manifold (1) obeying these equations will be said to be of the Einstein type. "Euclidean space" is a special case characterized by the

vanishing of the curvature tensor (4). More exactly we should here employ the term flat space since we must include the affine-euclidean as well as the ordinary euclidean species. Flat space thus means that the potentials g_{ik} in (1) can be reduced to constants, or that there is no permanent gravitation.

§ 2. EUCLIDEAN OR FLAT SPACE.

We now show that the light equation of an Einstein field cannot reduce to the simple form $dt^2 - dx^2 - dy^2 - dz^2 = 0$ *unless there is no permanent gravitation.*

For this purpose we use the more symmetric form

$$dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 = 0,$$

and prove that the curvature tensor necessarily vanishes.

We proceed to calculate the gravitational equations $G_{\mu\nu} = 0$ in the case where the manifold is of the form

$$ds^2 = \lambda(dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2), \tag{7}$$

where λ is an unknown function of the four coördinates. The potentials are here

$$g_{11} = g_{22} = g_{33} = g_{44} = \lambda,$$

the remaining six, g_{12}, g_{13}, etc., all vanishing. The determinant g reduces to λ^4; therefore

$$g^{11} = g^{22} = g^{33} = g^{44} = \lambda^{-1},$$

the other six, g^{12}, etc., also vanishing.

We thus find immediately the first symbols

$$[11, 1] = \tfrac{1}{2}\lambda_1, \qquad [11, 2] = -\tfrac{1}{2}\lambda_2,$$
$$[12, 1] = \tfrac{1}{2}\lambda_2, \qquad [12, 3] = 0,$$

where subscripts denote partial derivatives; and the second symbols

$$\begin{aligned} \{11, 1\} &= N_1, \qquad \{11, 2\} = -N_2, \\ \{12, 1\} &= N_2, \qquad \{12, 3\} = \quad 0 \end{aligned} \tag{8}$$

where, for convenience, we introduce

$$N = \tfrac{1}{2}\log\lambda. \tag{8'}$$

The function L occurring in the Einstein equations is then

$$L = \log\lambda^2 = 2\log\lambda = 4N.$$

Since our problem is symmetric in the four coördinates it will be sufficient to calculate G_{12} (case of unlike subscripts) and G_{11}, (case of like subscripts).

$$G_{12} = \{1\alpha, \beta\}\{2\beta, \alpha\} - \{12, \alpha\}_\alpha + L_{12} - \{12, \alpha\}L_\alpha.$$

The first term on the right is a summation of 16 terms, of which 10 vanish and 6 reduce each to N_1N_2; the second term is a sum of 4 terms contributing $-2N_{12}$; the third term is directly $4N_{12}$; the last term is a sum of 4, contributing $-8N_1N_2$. We have then

$$\begin{aligned} G_{12} &= 6N_1N_2 - 2N_{12} + 4N_{12} - 8N_1N_2 \\ &= 2N_{12} - 2N_1N_2. \end{aligned}$$

The like subscript case is a little more complicated:

$$\begin{aligned} G_{11} &= \{1\alpha, \beta\}\{1\beta, \alpha\} - \{11, \alpha\}_\alpha + L_{11} - \{11, \alpha\}L_\alpha \\ &= 4N_1^2 - N_2^2 - N_3^2 - N_4^2 - N_{11} + N_{22} + N_{33} + N_{44} \\ &\quad + 4N_{11} - 4N_1^2 + 4N_2^2 + 4N_3^2 + 4N_4^2 \\ &= 3N_{11} + N_{22} + N_{33} + N_{44} + 2(N_2^2 + N_3^2 + N_4^2). \end{aligned}$$

Hence the conditions (necessary and sufficient) that a manifold of the form

$$ds^2 = \lambda(dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2)$$

shall be an Einstein manifold (that is, $G_{\mu\nu} = 0$) are

$$\begin{aligned} &N_{12} - N_1N_2 = 0, \qquad \text{etc.}, \\ &3N_{11} + N_{22} + N_{33} + N_{44} + 2(N_2^2 + N_3^2 + N_4^2) = 0, \qquad \text{etc.} \end{aligned} \tag{9}$$

This is a set of 10 partial differential equations of the second order in one unknown function $N = \frac{1}{2}\log\lambda$.

It remains to show that the manifolds thus obtained are euclidean. This can be done either by actually integrating the above set, or by proving that the curvature tensor vanishes.

The easiest way to integrate the set is to use the transformation

$$N = -\log M. \tag{10'}$$

This gives

$$N_1 = -\frac{M_1}{M}, \qquad N_{12} = \frac{M_1M_2 - MM_{12}}{M^2}, \qquad N_{11} = \frac{M_1^2 - MM_{11}}{M^2}.$$

Hence the transformed set is

$$\begin{aligned} &M_{12} = 0, \qquad M_{13} = 0, \qquad \text{etc.} \\ &M_{11} = M_{22} = M_{33} = M_{44}. \end{aligned} \tag{10}$$

We easily find the general solution to be

$$M = a(x_1^2 + x_2^2 + x_3^2 + x_4^2) + a_1x_1 + a_1x_2 + a_3x_3 + a_4x_4 + a_5 \tag{11}$$

with the condition

$$a_1^2 + a_2^2 + a_3^2 + a_4^2 - 4aa_5 = 0, \tag{11'}$$

so that the result involves five arbitrary constants.

Since $N = \frac{1}{2}\log\lambda$, it follows that $\lambda = M^{-2}$; hence our manifold can be written

$$ds^2 = \frac{dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2}{M^2}. \tag{12}$$

This can be reduced by linear transformation of the coördinates to one of the three forms

$$\frac{\Sigma dx_i^2}{(\Sigma x_i^2)^2}, \qquad \frac{\Sigma dx_i^2}{(x_1 + ix_2)^2}, \qquad \Sigma dx_i^2, \tag{13}$$

which are found to be flat manifolds.

In second method we avoid the integration of set (9); and calculate instead the uncontracted curvature tensor $\{\alpha\beta, \gamma\delta\}$. Of these symbols it is sufficient to consider the types

$$\begin{array}{lll} \{11,.12\}, & \{12, 21\}, & \{11, 23\}, \\ \{12, 31\}, & \{21, 31\}, & \{12, 34\}. \end{array} \tag{14}$$

The first, third, and sixth vanish identically. The others give conditions of the types

$$\begin{gathered} N_{12} - N_1N_2 = 0, \qquad \text{etc.} \\ N_{11} + N_{22} + N_3^2 + N_4^2 = 0, \qquad \text{etc.} \end{gathered} \tag{15}$$

These are the conditions (necessary and sufficient) that

$$ds^2 = \lambda(dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2), \qquad N = \tfrac{1}{2}\log\lambda,$$

be euclidean.

It is obvious a priori that when this set is fulfilled so is the former set (9), since of course euclidean space obeys Einstein's equations. It is not obvious that the converse also holds; but it is a fact since we can verify directly that each set of ten equations in N is linearly reducible to the other set.

Hence the only manifolds of the form (7) *which obey Einstein's gravitational equations are (affine) euclidean.*

We may state this in more geometric form by noting that the only four-dimensional spreads which can be conformally represented on four-dimensional flat space are precisely those of the above form.

Hence of all the four-dimensional spaces which are conformally representable on flat space, the only ones which are of the Einstein type are (affine) euclidean.

This means that when a conformal representation of an Einstein manifold on a flat space is possible, the manifold is isometric to flat space.

§ 3. NEARLY-EUCLIDEAN MANIFOLDS.

We proceed to generalize our result to curved Einstein spaces; namely, to show that the light equation determines the space, or that two conformally equivalent Einstein spaces are applicable. We shall here give the proof for the case where the Einstein spaces differ infinitesimally from flat space.

By a *nearly-euclidean* (or approximately flat) space we understand

$$ds^2 = \Sigma dx_i^2 + \Sigma h_{ik} dx_i dx_k, \tag{16}$$

where the h's denote 10 functions (of the coördinates x_1, x_2, x_3, x_4), which functions together with their derivatives are supposed to be small, so that their squares and products are neglected. We could write instead

$$\Sigma dx_i^2 + \epsilon \Sigma h_{ik} dx_i dx_k,$$

where ϵ is a small number and the h's are arbitrary finite functions; but the first notation is more convenient.

The curvature tensor and the Einstein tensor then reduce to *linear* differential expressions of the second order. Of the 10 Einstein equations it is sufficient to write those corresponding to G_{12} and G_{11}. These are (the last two subscripts denoting partial differentiation)

$$\begin{aligned} &h_{13,\,23} + h_{23,\,13} - h_{12,\,33} + h_{14,\,24} + h_{24,\,14} - h_{12,\,44} \\ &\qquad\qquad - h_{33,\,12} - h_{44,\,12} = 0, \\ &2(h_{12,\,12} + h_{13,\,13} + h_{14,\,14}) - h_{11,\,22} - h_{11,\,33} - h_{11,\,44} - h_{22,\,11} \\ &\qquad\qquad - h_{33,\,11} - h_{44,\,11} = 0. \end{aligned} \tag{17}$$

In deriving these it is sufficient to note that for the above ds^2 we have

$$\begin{aligned} g_{11} &= 1 + h_{11}, & g_{12} &= h_{12}, \\ g &= 1 + \Sigma h_{ii}, & L &= \tfrac{1}{2}\Sigma h_{ii}, \\ g^{11} &= 1 - h_{11}, & g^{12} &= -h_{12}, \end{aligned} \tag{17'}$$

with the three-index symbols (both kinds here have same values)

$$\begin{aligned} \{11, 1\} &= \tfrac{1}{2}h_{11,\,1}, & \{11, 2\} &= h_{12,\,1} - \tfrac{1}{2}h_{11,\,2}, \\ \{12, 1\} &= \tfrac{1}{2}h_{11,\,2}, & \{12, 3\} &= \tfrac{1}{2}(h_{13,\,2} + h_{23,\,1} - h_{12,\,3}). \end{aligned} \tag{17''}$$

§ 4. THE LIGHT EQUATION OF A NEARLY-EUCLIDEAN ds^2.

This is found by putting ds^2 equal to zero; that is

$$\Sigma dx_i^2 + \Sigma h_{ik} dx_i dx_k = 0. \tag{18}$$

If a second manifold determined by 10 new functions H_{ik} is to have the same light equation, that is, if

$$\Sigma dx_i^2 + \Sigma H_{ik} dx_i dx_k = 0 \tag{19}$$

is to differ from the former equation merely by a factor λ, this factor must obviously differ only slightly from unity, that is,

$$\lambda = 1 + \mu,$$

where μ is a small function (of the same order as the h's). This shows that

$$\begin{aligned} H_{ii} &= h_{ii} + \mu \qquad (i = 1, 2, 3, 4), \\ H_{ij} &= h_{ij} \qquad (i \neq j). \end{aligned} \tag{20}$$

Suppose now that both spaces are of the Einstein type. Our problem is to find the conditions on the function μ which follow from the fact that the functions h and also the functions H obey the linear equations of second order (17). Substituting and subtracting we find these 10 equations

$$\begin{aligned} \mu_{12} &= 0, \qquad \text{etc.}, \\ 3\mu_{11} + \mu_{22} + \mu_{33} + \mu_{44} &= 0, \qquad \text{etc.} \end{aligned} \tag{21}$$

The last four equations show that

$$\mu_{11} = \mu_{22} = \mu_{33} = \mu_{44} = 0; \tag{21'}$$

hence all the second derivatives of μ vanish. It follows that the function is linear, say

$$\mu = a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4 + b. \tag{22}$$

Our result is that if ds^2 defines any given nearly-euclidean Einstein space, then the only other possible nearly-euclidean Einstein spaces which have the same light equation are defined by

$$dS^2 = (1 + \mu) ds^2, \tag{23}$$

where μ is a linear function (22).

It remains to show that these spaces are equivalent (isometric) to the original (18). To do this, consider the general infinitesimal point transformation

$$X_i = x_i + \xi_i, \tag{24}$$

where the ξ's denote small functions of the four coördinates.

We have

$$dX_i = dx_i + \xi_{i,j}dx_j \tag{24'}$$

(summation respect to j understood); therefore

$$\begin{aligned} \Sigma dX_i^2 &= \Sigma dx_i^2 + 2\xi_{i,k}dx_i dx_k \\ &= \Sigma dx_i^2 + 2\xi_{1,1}dx_1^2 + \text{etc.} + 2(\xi_{1,2} + \xi_{2,1})dx_1 dx_2 + \text{etc.} \end{aligned} \tag{24''}$$

We first inquire when this reduces to $(1 + \mu)\Sigma dx_i^2$. The conditions are

$$\begin{gathered} \xi_{1,1} = \xi_{2,2} = \xi_{3,3} = \xi_{4,4} = \mu, \\ \xi_{1,2} + \xi_{2,1} = 0, \quad \text{etc.} \end{gathered} \tag{25}$$

We find that with the above value (22) of μ this set of 10 equations for the 4 unknown functions is consistent, the complete solution being

$$\begin{aligned} \xi_1 &= x_1\Sigma a_i x_i - \tfrac{1}{2}a_1\Sigma x_i^2 + A_{12}x_2 + A_{13}x_3 + A_{14}x_4 + bx_1 + B_1, \\ \xi_2 &= x_2\Sigma a_i x_i - \tfrac{1}{2}a_2\Sigma x_i^2 + A_{21}x_1 + A_{23}x_3 + A_{24}x_4 + bx_2 + B_2, \\ \xi_3 &= \\ \xi_4 &= \end{aligned} \tag{26}$$

where $A_{21} + A_{12} = 0$, etc., so that the result depends on 15 arbitrary constants. This is recognized as the general infinitesimal conformal transformations in flat space of four dimensions.

Apply this transformation to (16). The result, neglecting terms containing products of h's and ξ's, is found to be

$$(1 + \mu)\Sigma dx_i^2 + \Sigma h_{ik}dx_i dx_k;$$

which may be rewritten, in virtue of (20),

$$\Sigma dx_i^2 + \Sigma H_{ik}dx_i dx_k.$$

This shows that the two quadratic forms, one determined by the h's, the other by the H's of (20) are equivalent when μ has the linear form (22). We may therefore state the

THEOREM. *If two nearly-euclidean spaces both obey Einstein's equations and have the same light equation, they are necessarily equivalent (isometric).*

A geometric restatement would be:

If two nearly-euclidean spaces of the Einstein type are capable of conformal representation, they are necessarily isometric.

§ 5. DETERMINATION BY QUADRATURES.

Suppose we are given the light equation of some unknown Einstein space; how shall we find that space. *This can always be done by differentiations*

and quadratures. For suppose that the ten functions h_{ik} in the equation (18) do not satisfy, as they are given, the linear gravitational equations (17); the problem is then to find a function μ so that the ten functions H_{ik} found by (20) shall satisfy (17). The ten equations in μ thus obtained can be solved for the second derivatives μ_{ik}, which are thus expressed as known functions of the x's. The equations, by assumption consistent, can therefore be solved by quadratures. The result is determined up to an additive linear term which does not essentially affect the quadratic form ds^2 obtained.

§ 6. GENERAL EINSTEIN MANIFOLDS.

If we do not assume our manifolds to be nearly-euclidean, the discussion is of course more difficult since we have then to face the exact non-linear expressions (6). We shall confine ourselves here to the statement that we are lead to a set of ten non-linear equations of the second order for the unknown factor λ; these can be solved for the ten second derivatives in terms of the first derivatives; existence theorems then show that *the solution depends on not more than five arbitrary constants* (one of these is trivial, being merely a constant factor). In the special case of § 2 it was easy to show that the ∞^5 quadratic forms actually obtained were all equivalent. The general discussion will be left for a later paper.

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NOTE ON EINSTEIN'S EQUATION OF AN ORBIT.

By F. Morley.

The differential equation of an orbit is given in Eddington's "Report on the Relativity Theory of Gravitation"* as

$$(1) \qquad \left[\frac{d}{d\varphi}\frac{1}{r}\right]^2 = 2m/r^3 - 1/r^2 + 2m/h^2r - (1-c^2)/h^2,$$

r and φ being the polar coördinates of the planet. Forsyth† has discussed the equation in connection with elliptic functions. The matter is of so great interest that a second elliptic handling seems worth publication.

We have, from Eddington,

$$c^2 = 1 - m/a, \qquad h^2 = ma(1-e^2),$$

where a is the "major semi-axis" and e the eccentricity; therefore

$$1 - c^2 = m^2(1-e^2)/h^2.$$

The number m^2/h^2 is, in Eddington's units, about 10^{-8}. Call it α, so that

$$1 - c^2 = \alpha(1-e^2).$$

In the calculations α^2 will be neglected when lower powers are present.

Writing $x = m/r$, the equation becomes

$$(2) \qquad \left(\frac{dx}{d\varphi}\right)^2 = 2x^3 - x^2 + 2\alpha x - \alpha(1-e^2).$$

The roots of the cubic on the right are $\frac{1}{2}$, 0, 0 when $\alpha = 0$. Hence writing $x = \frac{1}{2} + k_1\alpha, k_2\alpha, k_3\alpha$ and determining the coefficients k, the roots are

$$x_1 = \tfrac{1}{2} + 2\alpha, \qquad x_2 = \alpha(1-e), \qquad x_3 = \alpha(1+e).$$

To bring equation (2) into Weierstrass's form

$$\left(\frac{dp}{dv}\right)^2 = 4p^3 - g_2p - g_3$$

we write $d\varphi = \sqrt{2}dv$, $x = p + 1/6$. The invariants g_2 and g_3 are for a quartic with coefficients a, b, c, d, e

$$ae - 4bd + 3c^2$$

* 2d Edition (1920), p. 50. Einstein, Berlin Sitzungsberichte, 1915, p. 831.

† Proc. Royal Society, series A, vol. 97, April, 1920.

and

$$\begin{vmatrix} a & b & c \\ b & c & d \\ c & d & e \end{vmatrix}.$$

Here a, b, c, d, e are respectively 0, 1, $-\frac{1}{3}$, α, $-2\alpha^2(1-e^2)$, so that, accurately,

$$g_2 = \tfrac{1}{3}(1 - 12\alpha), \tag{3}$$

$$g_3 = \tfrac{1}{2}\tfrac{1}{7}[1 - 18\alpha + 54\alpha^2(1-e^2)]. \tag{4}$$

Hence the discriminant Δ, or $g_2^3 - 27g_3^2$, is

$$\Delta = 4e^2\alpha^2 - 64\alpha^3. \tag{5}$$

The roots being real, the network or lattice which gives rise to the function p is rectangular. The essence of the problem is the determination of a period-rectangle. Let $2\omega_1$ be the real positive period, $2\lambda i\omega_1$ the imaginary period, and write as usual

$$q = e^{-\pi\lambda}.$$

The number λ is positive, and it may be called the shape of the rectangle.

The invariants g_2, g_3, are connected with the periods $2\omega_1$ and $2\omega_2$ by the equations

$$g_2 = 60 \sum{}' \frac{1}{(2m_1\omega_1 + 2m_2\omega_2)^4}, \qquad g_3 = 140 \sum{}' \frac{1}{(2m_1\omega_1 + 2m_2\omega_2)^6},$$

summed for all integer pairs m_1, m_2, the pair 0, 0 excluded. For calculation the double sums are expressed as single sums or products; and the appropriate formulæ are given in works on the elliptic functions.* But as the proofs of the full formulæ are necessarily complicated, I shall interpolate a proof of the approximate formulæ of a kind that is at once intelligible.

We have, n being any integer,

$$\sum_{-\infty}^{\infty} \frac{1}{(x+n)^2} = \frac{\pi^2}{\sin^2 \pi x}.$$

Differentiating,

$$\sum_{-\infty}^{\infty} \frac{1}{(x+n)^3} = \frac{\pi^3 \cos \pi x}{\sin^3 \pi x}.$$

Differentiating again,

$$\sum_{-\infty}^{\infty} \frac{1}{(x+n)^4} = \pi^4 \frac{\cos^2 \pi x + \frac{1}{3} \sin^2 \pi x}{\sin^4 \pi x} = \pi^4 \left[\frac{1}{\sin^4 \pi x} - \frac{2}{3 \sin^2 \pi x} \right].$$

This enables us to sum the series for g_2, when m_2 is fixed. When $m_2 = 0$,

* They may be found in Harkness and Morley, "Theory of Functions," p. 321, formulæ 75 and 76, and p. 324, formula 82.

we get the first tier

$$\frac{2\cdot 60}{(2\omega_1)^4}\left[1+\frac{1}{2^4}+\frac{1}{3^4}+\cdots\right]=\frac{15}{2\omega_1^4}\cdot\frac{\pi^4}{90}=\frac{1}{12}\left(\frac{\pi}{\omega_1}\right)^4.$$

When $m_2 = 1$, we get the second tier

$$\sum_n \frac{1}{(2\omega_2+2m_1\omega_1)^4}=\frac{1}{(2\omega_1)^4}\sum\frac{1}{(i\lambda+m_1)^4}$$

$$=\left(\frac{\pi}{2\omega_1}\right)^4\left(\frac{1}{\sin^4\pi i\lambda}-\frac{2}{3\sin^2\pi i\lambda}\right).$$

Now

$$\sin\pi i\lambda=\left(\frac{e^{-\pi\lambda}-e^{\pi\lambda}}{2i}\right)=\left(q-\frac{1}{q}\right)\Big/2i.$$

Hence the tier $m_2 = 1$ gives

$$60\left(\frac{\pi}{2\omega_1}\right)^4\left[\frac{16q^4}{(1-q^2)^4}+\frac{8q^2}{3(1-q^2)}\right]$$

or, neglecting q^4,

$$10\left(\frac{\pi}{\omega_1}\right)^4 q^2.$$

For the tier $m_2 = -1$ we get the same result. For the tiers $m_2 \equiv \pm 2$ we have to replace q by q^2 and so on. Hence the formula

$$\left(\frac{\omega_1}{\pi}\right)^4 g_2 = \tfrac{1}{12} + 20q^2 + \cdots. \tag{6}$$

A similar argument, with further differentiation, gives

$$\left(\frac{\omega_1}{\pi}\right)^6 g_3 = \tfrac{1}{216} - \tfrac{7}{3}\,q^2 + \cdots \tag{7}$$

and from these

$$\left(\frac{\omega_1}{\pi}\right)^{12}\Delta = q^2 + \cdots. \tag{8}$$

From the formulæ (6) and (8) we have approximately

$$\Delta/g_2^3 \doteq 12^3 q^2$$

and on the other hand from (3) and (5) we have

$$\Delta/g_2^3 \doteq 4\cdot 27\cdot e^2\alpha^2.$$

Hence approximately

$$q^2 \doteq e^2\alpha^2/4^2,$$

$$q \doteq e\alpha/4.$$

For Mercury the eccentricity e is approximately $1/5$, so that

$$q \doteq 10^{-8}/20.$$

Calculating $\pi\lambda = \log 1/q = 9 \log 10 + \log 2$, the shape λ of the rectangle of the periods turns out to be nearly 7. As v travels round the rectangle of half periods, p takes all real values, so that the radius vector

$$r = m/x = m/(p + \tfrac{1}{6})$$

also takes all real values. For the actual orbit we take v on the upper side of the rectangle, from $v = \omega_1 + \omega_2$, $p = \alpha(1 + e)$, corresponding to perihelion, to $v = \omega_2$, $p = \alpha(1 - e)$ or aphelion.

Finally, the corresponding change in φ is $\varphi = \sqrt{2}\omega_1$, where

$$\left(\frac{\omega_1}{\pi}\right)^4 g_2 = \tfrac{1}{12}$$

for q^2, being of order α^2, is here negligible. But from (3)

$$g_2 = \tfrac{1}{3}(1 - 12\alpha).$$

Hence

$$\frac{1}{4}\left(\frac{\varphi}{\pi}\right)^4 \cdot \frac{1}{3}(1 - 12\alpha) = \frac{1}{12}.$$

$$\left(\frac{\varphi}{\pi}\right)^4 = 1/(1 - 12\alpha) = 1 + 12\alpha,$$

$$\frac{\varphi}{\pi} = 1 + 3\alpha.$$

So that in passing from perihelion to aphelion the angle passed through is not π, but

$$\pi(1 + 3m^2/h^2).$$

A ONE-TO-ONE REPRESENTATION OF GEODESICS ON A SURFACE OF NEGATIVE CURVATURE.

By Harold Marston Morse.

Introduction.

Surfaces of negative curvature and geodesics upon such surfaces have been considered by J. Hadamard in the paper,* cited below. The paper by Hadamard is in two parts, in the first of which he establishes the existence of a very general class of surfaces of negative curvature; the remainder of Hadamard's paper is devoted to considerations of geodesics upon such surfaces.

In the present paper only those geodesics on the given surfaces of negative curvature are considered which, if continued indefinitely in either sense, lie wholly in a finite portion of space. A class of curves is introduced each of which consists of an unending succession of the curve segments by which the given surface, when rendered simply connected, is bounded. It is shown how a curve of this class can be chosen so as to uniquely characterize some geodesic lying wholly in a finite portion of space. Conversely, it is shown that every geodesic lying wholly in a finite portion of space, is uniquely characterized by some curve of the above class. Use is made of this representation of the geodesics considered to prove several fundamental theorems.

The results of this paper and the representation of geodesics obtained here, will be used in a later paper to establish the existence of a class of geodesics called recurrent geodesics of the discontinuous type.† This class of geodesics offers the first proof that has been given in the general theory of dynamical systems, of the existence of recurrent motions of the discontinuous type.

Part I.

The Surface.

§ 1. We will consider surfaces without singularities in finite space. We will suppose the surface divisible into overlapping regions such that every point of the surface lying in a finite portion of space is contained as an interior point in some one of a finite number of these regions, and such that

* Les surfaces à courbures opposées et leur lignes geodesiques," Liouville, *Journal de Mathématique*, 5 Sér., t. 4, p. 27, 1898.

† G. D. Birkhoff, "Quelques théorèmes sur le mouvement des systèmes dynamiques," *Bull. de la Soc. Math. de France*, Vol. 40, p. 303, 1912.

the Cartesian coördinates x, y, z, of the points of any one of these regions can be expressed in terms of two parameters, u and v, by means of functions with continuous derivatives up to a convenient order, at least the third, and such that

$$\left(\frac{D(xy)}{D(uv)}\right)^2 + \left(\frac{D(xz)}{D(uv)}\right)^2 + \left(\frac{D(yz)}{D(uv)}\right)^2 \neq 0.$$

By a *curve* on the surface we will understand any set of points on the surface in continuous correspondence with the points of an interval on a straight line, including one, both, or neither of its end points.

We will suppose the Gaussian curvature of the surface to be negative at every point, with the possible exception of a finite number of points, at which points the curvature will necessarily be zero. A first result, given by Hadamard in the paper already referred to, is that a surface of negative curvature cannot be contained in any finite portion of space.

§ 2. By a *funnel* of a surface will be meant a portion of a surface topographically equivalent to either one of the two surfaces obtained by cutting an unbounded circular cylinder by a plane perpendicular to its axis. We will consider surfaces of negative curvature whose points outside of a sufficiently large sphere with center at the origin consist of a finite number of funnels. Each of these funnels will be cut off from the rest of the surface along a simple closed curve. These curves will be taken sufficiently remote on the funnels to be entirely distinct from one another.

An unparted hyperboloid of revolution is an example of a surface of negative curvature with two funnels.

From the definition of a funnel, it follows that by a continuous deformation of the closed curve forming the boundary of the funnel, the funnel may be swept out in such a way that every point of the funnel is reached once and only once. Hadamard considers two classes of funnels: those which can be swept out by closed curves which remain less in length than some fixed quantity, and those which do not possess this property. Surfaces with funnels of the first sort are for several reasons of less general interest than those with funnels of the second sort. In the present paper surfaces with funnels only of the second sort will be considered. Hadamard showed that there exist surfaces of negative curvature possessing funnels all of the second sort of any arbitrary number exceeding one, and such that the surface obtained by cutting off these funnels is of an arbitrary genus.

§ 3. We shall consider surfaces which possess at least two funnels of the second sort, and of the surfaces with just two funnels of the second sort we will exclude those surfaces that are topographically equivalent to an unbounded circular cylinder. Hadamard proves that on the surfaces we

are considering there exists one and only one closed geodesic that is deformable into the boundary of a given funnel, and that this geodesic possesses no multiple points, and no points in common with the other closed geodesics that are deformable into the boundaries of the other funnels.

We shall denote these closed geodesics, say v in number, by

$$g_1, g_2, \cdots, g_v. \tag{1}$$

They will form the complete boundary of a part of the surface, contained in a finite part of space. We denote this bounded surface by S. It may be proved as a consequence of the assumptions concerning the surface, made in § 1, that S is of finite connectivity. According to the well known theory, S may be rendered simply connected as follows: We first cut S along a system of curves,

$$h_1, h_2, \cdots, h_{v-1},$$

each of which has one end point on an arbitrarily chosen point, P on g_v, and the other, respectively, on the geodesic,

$$g_1, g_2, \cdots, g_{v-1}$$

with the same subscript; and no two of which have a point other than P in common, and no points other than their end points in common with the geodesics of the set (1). There then results a surface with a single boundary. This surface can accordingly be rendered simply connected by $2p$ curves,

$$c_1, c_2, \cdots, c_{2p},$$

each of which curves can be taken as beginning and ending at P, and having no other points than P in common with any of the other curves or geodesics.

We denote by T, the simply connected piece of surface obtained by cutting S along the above curves. It may readily be proved as a consequence of the assumptions made concerning the representation of the given surface, that T is topographically equivalent to a plane region consisting of the interior and boundary points of a circle.

The Infinitely Leaved Surface M.

§ 4. We suppose an infinite number of distinct copies of T spread out above the surface S, in such a manner that points arising from a given point of S overhang the given point of S. We denote by

$$A_i^+, \qquad i = 1, 2, \cdots, 2p + v - 1.$$

that boundary piece of T which arises from an arbitrarily chosen side of the ith one of the $2p + v - 1$ cuts,

$$c_1, c_2, \cdots, c_{2p}; \qquad h_1, h_2, \cdots, h_{v-1}, \tag{1}$$

which we have made in S, and by A_i^- that boundary piece of T that arises from the opposite side of the same cut.

Consider now a first copy of T, say M_1, and a particular boundary piece of M_1, say A_i^+. We join M_1 to a second copy of T, joining A_i^+ of M_1 to A_i^- of the second copy of T. In the same manner, we join to M_1 at each different piece of the remaining boundary pieces, a different copy of T, joining each two copies of T along two boundary pieces arising from opposite sides of the same cut. We denote the bounded surface so obtained by M_2.

Proceeding in the same manner, we join to each different boundary piece of M_2 a different copy of T, and one not already a part of M_2, obtaining thereby a new surface M_3. From M_3 we form a new surface M_4, as we formed M_3 from M_2, and continue this process indefinitely, obtaining an enumerable infinity of surfaces each of which contains the preceding. We will understand two different copies of T making up M_n, to have a point in common, if and only if we have expressly joined the two copies together at this point.

§ 5. DEFINITION. *By the surface M we will understand that infinitely leaved surface spread over S that contains each of the surfaces M_n, and every point of which is contained in some one of the surfaces M_n.*

The boundaries of M will consist of those boundary pieces, of the copies of T that make M up, that arise from the geodesics,

$$g_1, g_2, \cdots, g_v.$$

We have made the cuts in S so that on each of the boundary pieces,

$$g_1, g_2, \cdots, g_{v-1}, \tag{2}$$

lies one end point of just one cut. The remaining $4p + v - 1$ end points of the cuts lie at a single point on g_v. On M a sufficiently restricted neighborhood of any one of the points on the geodesics (2), at which lies an end point of a cut, is, when severed along this cut, best described as topographically equivalent to the neighborhood of a point on the boundary of a half plane, when that half plane is severed along a ray distinct from the boundary of the half plane, and issuing from the given point. Similarly a sufficiently restricted neighborhood of that point on g_v at which lie the remaining $4p + v - 1$ end points of cuts, is, when severed along these cuts, topographically equivalent to the neighborhood of a point on the boundary of a half plane, when that half plane is severed along $4p + v - 1$ rays, distinct from each other and from the boundary of the half plane, and issuing from the given point. A sufficiently restricted neighborhood of any other point of M, is topographically equivalent to the complete neighborhood of any point in the plane.

We may conclude that *a sufficiently restricted neighborhood of any point on M is a one-leaved copy of the neighborhood on S of the point overhung on S.*

§ 6. It will be convenient to denote by R a plane region consisting of the points on and interior to a circle.

Any single copy of T is topographically equivalent to a region R. From the method of formation of the surfaces M_n of § 4, it follows that any one of the surfaces M_n is topographically equivalent to a region R. Now from the definition of M in § 5, it follows that any set of points on M that lie in only a finite number of copies of T making up M, will be contained in some one of the surfaces M_n forming a part of M. *Hence any set of points that lie in only a finite number of the copies of T that make up M, will be contained in some region of M, topographically equivalent to a region R.*

§ 7. An infinite set of points on M will be said to have a *limit point P*, if there exists an infinite number of points of the set in every neighborhood of P, on M.

We will prove that a set of points every infinite subset of which has at least one limit point on M, cannot have points in more than a finite number of different copies of T making up M.

If there were points of the set in more than a finite number of the copies of T, there would exist a subset of the given set, say Q, containing an infinite number of points of the given set, and such that each different point would lie in a different copy of T. By the hypothesis made concerning the given set, Q has at least one limit point on M, say P. There are accordingly an infinite number of points of Q in every neighborhood of P, and since no two points of Q lie in the same copy of T, there must be an infinite number of copies of T with points in the neighborhood of P. This is impossible; for all the points in a sufficiently restricted neighborhood of a point on M lie on at most $4p + v$ copies of T, that are joined together at that point.

Hence the result required is proved. Combining this result with the conclusion of the preceding section, we have that *a set of points on M every infinite subset of which has at least one limit point on M, can be included in a region of M, made up of a finite number of copies of T, and topographically equivalent to the plane region consisting of the interior and boundary points of a circle.*

Curves on M Overhanging Curves on S.

§ 8. DEFINITION. Let there be given on S a curve k, and on k a point P. Let P' be any point on M, overhanging P. Starting from a particular point of k, say Q, let P trace out k, first in one sense, and then starting again at Q, in the other sense. As P moves continuously on S, let P' move continuously on M, always overhanging P, and starting at both the times that P starts at Q, at the same point, a point, say Q', overhanging Q. The

continuous curve traced out by P' on M, will be said to *correspond* to k on S.

It follows from the result of § 5, that the curve traced out by P', is uniquely determined by k and the choice of the points Q and Q'.

DEFINITION. Let h be a curve on S that is closed. Let h be cut at some point so as to form a curve segment k. Let h' be any curve segment on M corresponding, in the sense of the preceding definition, to k. h and h' will be said to *correspond*.

Every curve on M obviously corresponds to one and only one curve on S, while a given curve on S will correspond to an infinite number of different curves on M.

§ 9. *If a curve on S is both closed and deformable into a point, it corresponds on M only to curves that are closed.*

Let h be a curve on S that is both closed and deformable into a point. Let Q' and P' be the end points of a curve segment on M corresponding in the sense of the preceding definition, to h. Since h is closed, Q' and P' overhang a single point on h, say Q.

Suppose now that the above lemma were not true, and that Q' and P' were distinct points on M. Let h be continuously deformed into a point. At the same time let the curve segment joining Q' to P', say h', be continuously deformed on M in such a manner that h' corresponds at each stage of the deformation to the curve into which h has been deformed at that stage, while Q' and P' overhang Q. When, at the end of this process, h reduces to a point, h' must also reduce to a point.

On the other hand P' and Q', the end points of h', in their initial positions were supposed to be distinct points. Since P' and Q' move continuously throughout the deformation, there must then be a first position on M, say P'', at which they are coincident. P' and Q', throughout this deformation, overhung one point of S, namely Q. Hence the point P'' will have in every neighborhood on M pairs of points overhanging the same point of S. This, however, is contrary to the result of § 5, from which contradiction we infer the truth of the above lemma.

§ 10. The following lemma serves as the converse of the preceding.

If a curve on M is closed, it corresponds on S to a curve that is both closed and deformable into a point.

Denote by h the given closed curve on M. It follows from § 7 that there exists a region of M, say M', that contains all the points of h as interior points, and is topographically equivalent to a plane region consisting of the interior and boundary points of a circle. On M', h can be deformed into a point, and hence on M.

Denote by F a continuous family of closed curves on M through the mediation of which h may be deformed into a point on M. On S the

closed curve corresponding to h, may be deformed into a point through the mediation of that continuous family of closed curves which correspond to the family F on M.

§ 11. DEFINITION. Let r be any integer, positive, negative, or zero. Let T_r denote a particular copy of T of M. By a *linear set* of copies of T of M, will be understood a region of M consisting of a set of the copies of T in M of the form,

$$(1) \qquad \cdots T_{-2}, T_{-1}, T_0, T_1, T_2, \cdots,$$

or of the form of any subset of consecutive symbols of (1), in which any three successive copies of T are three different copies of T of M, and in which any two successive copies of T are copies of T that in M are joined along a common boundary piece. A linear set which has no first or last copy of T, will be termed an *unending* linear set.

A linear set is of the nature of an unclosed ribbon of surface. That a linear set does not form a multiple-covered region of M will now be proved.

(*A*) *No copy of T of M appears more than once in any linear set.*

Starting with any copy of T, say T^0, of a given linear set, if possible, let T' be the first successor of T^0 that is identical with some copy of T of the given linear set between T^0 and T', say T''. That linear subset of the given linear set that consists of T'' and its successors up to, but not including T', forms a simply covered, connected region of M, in which the two sides of any boundary piece common to two of its successive copies of T, can be joined by a curve that does not cross that boundary piece. This is im. possible, since each boundary piece common to two copies of T of M joins two boundary points of M and divides M into two regions. Cf. § 5 and § 6.

(*B*) *If T' and T'' are any two copies of T of the copies of T that make up M, there exists one and only one linear set of which T' is the first, and T'' the last member.*

We form a linear set in which the first copy of T is T' and in which the successor of T', say T_1, is that copy of T that adjoins T along that boundary piece of T' that separates T' from that part of the surface M that contains T''. If T_1 is not T'', its successor shall be that copy of T that adjoins T_1 along that boundary piece of T_1 that separates T_1 from that part of the surface in which T'' lies. We continue this process indefinitely, or until T'' appears in the linear set formed.

Join any interior point of T' to an interior point of T'' by a curve, say k, lying wholly on M. Each copy of T determined in the process of the preceding paragraph contains at least a point of k. But from the result of § 7 we infer that there are only a finite number of copies of T of M that contain any point of k. It follows that the process of the preceding paragraph will lead after a finite number of steps to T''.

There is but one linear set of which T' is the first and T'' the last copy of T. For no such linear set can contain as a part of its boundary that boundary piece of T' that separates T' from that part of the surface that contains T''. Hence, in any such linear set T' and T_1 must appear at least once as successive copies of T. But according to the result (A), no copy of T of M appears more than once in any linear set; it follows that T_1 is the successor of T' in any linear set joining T' to T''.

In a similar manner it follows that any linear set joining T' to T'', contains T_2 as the successor to T_1. Continuing this method of proof, it is seen that every linear set joining T' to T'', is identical with the first such linear set that we have formed.

§ 12. We denote by H, the set of all curve segments on M each of which corresponds on the original surface to some one of the segments,

$$c_1, c_2, \cdots, c_{2p}; \qquad h_1, h_2, \cdots, h_{v-1}; \qquad g_1, g_2, \cdots, g_v.$$

Every boundary piece of the copies of T of M, is included in the set H.

DEFINITION. Let r be any integer, positive, negative, or zero. Let k_r be a particular piece of the set H of M. By a *reduced curve* of the set H will be understood a continuous curve composed of a set of pieces of the set H of the form,

$$\cdots, k_{-2}, k_{-1}, k_0, k_1, k_2, \cdots, \tag{1}$$

or of the form of any subset of consecutive symbols of (1), in which no two consecutive pieces are copies of the same piece of the set H, and which contains no pieces corresponding to g_v on the original surface. A reduced curve without end points will be called an *unending* reduced curve.

(A) *No reduced curve can begin and end at the same point.*

Starting with any point, say P, of a given reduced curve, if possible, let P' be the first point following P that is identical with some point of the reduced curve between P and P', say P''. The segment $P'P''$ of the given reduced curve forms a closed curve without multiple points, which closed curve we denote by C.

According to the result of section 7, C can be included in some simply connected region of M consisting of a finite number of copies of T. C, accordingly, forms the boundary of a simply connected region of M, say R, all of whose points are contained in a finite number of copies of T of M. Since C contains no points interior to any copy of T, R contains each copy of T of which it contains any interior points.

Taking the copies of T in the order in which they were added in section 4 to form M, let T' be the last copy of T of R. T', like every other copy of T of M, is joined along a common boundary piece to but one copy of T of

M, say T'', that was added to M prior to T'. R accordingly contains none of the copies of T that are adjoined to T' along a common boundary piece except T''. Hence all of the boundary of T', except the piece common to T' and T'', must form part of the boundary of R, and hence a part of C. In particular, C contains that piece of the boundary of T' that corresponds on the original surface to the boundary geodesic g_v, contrary to the definition of a reduced curve. From this contradiction we infer the truth of the lemma.

The following result is an immediate consequence of the result (A).

(B) *A reduced curve can have no multiple points.*

(C) *There is one and only one reduced curve joining any two given points on pieces of the set* H.

Denote the two given points by P and Q. Among the curves consisting of any continuous succession of pieces of the set H, there obviously exists a variety of curves joining P to Q. If in any such curve, each piece that corresponds on the original surface to g_v, be replaced by the remainder of the boundary of the copy of T containing that piece, there will then result a curve containing no piece corresponding to g_v on the original surface. Of all curves of the latter class joining P to Q, any one that contains the minimum number of pieces of the set H, will be such that no two successive pieces are copies of the same piece of the set H, and will accordingly be a reduced curve.

If possible, let h' and h'' be two different reduced curves joining P to Q. Tracing out h' and h'' respectively, starting from P, let k' and k'' be the first pieces on h' and h'' respectively, that are different. That portion of h' that begins at Q, and ends with k', followed by that portion of h'' that begins with k'', and ends with Q, is a reduced curve with initial and final point at the same point, namely Q. This is contrary to the result (A) of this section, from which contradiction we infer the truth of the lemma (C).

(D) *A reduced curve never returns to a copy of* T *which it has once left.*

If this assertion be false, there exists some continuous segment of a reduced curve, say k, with end points, say P and Q, belonging to the same copy of T of M, and with no further points in common with that copy of T. A reduced curve cannot begin and end at the same point; hence P and Q are different points. However, let k' be that part of the boundary of the given copy of T that joins P to Q, and does not contain any piece of the set H corresponding to g_v on the original surface. k', taken with k, forms a segment of a reduced curve that may be considered as beginning and ending at P. We conclude that no such segment as k exists, and the lemma is proved.

§ 13. The following is the first of two theorems showing the relation of reduced curves to linear sets.

(*A*) *An unending linear set of copies of* T *contains one and only one unending reduced curve.*

Let T^0, T' and T'', be any three successive copies of T in the given linear set. There exist two distinct and continuous portions of the boundary of T', of which one may reduce to a point such that each joins some boundary point of T^0 to a boundary point of T'', without containing any other points of T^0 and T''. We associate with T' that one of these portions that does not contain a piece of the set H corresponding to g_v on the original surface. Denote this portion by k. The set of all curve segments such as k, taken in the order in which they arise from the given linear set, will not necessarily form a continuous curve. However by a proper addition of the boundary pieces common to successive copies of T of the given linear set, the resulting curve will be continuous. If these additions are made only when necessary to make the curve continuous, the resulting curve will also be a reduced curve.

If possible, let r be a second reduced curve contained in the given linear set. As proved in the preceding section, a reduced curve never returns to a copy of T which it has once left; it follows that r contains at least a point of each of the pieces of the set H that separate members of the given linear set, and hence contains each of the points or segments k, associated in the first paragraph of this proof with each copy of T of the given linear set. But according to the result (*C*) of section 12, there is but one segment of a reduced curve joining any two points of M. Hence any reduced curve that contains each of the points or curve segments k, associated with each copy of T of the given linear set, is thereby uniquely determined. There is thus but one reduced curve contained in the given linear set.

The preceding result has its converse in the following:

(*B*) *An unending reduced curve is contained in one and only one linear set, which set is an unending linear set.*

Let

$$\cdots, P_{-2}, P_{-1}, P_0, P_1, P_2, \cdots,$$

be a set of points which lie on the given reduced curve in the order of their subscripts, and which are such that any point of the given curve lies between P_{-n} and P_n for n a sufficiently large positive integer, and where P_{-1} and P_1 are supposed to be taken so as not to lie on a common copy of T of M. Let C_n be the segment joining P_{-n} to P_n of the given reduced curve. Let L_n be a linear set containing P_{-n} and P_n, and made up of the smallest possible number of copies of T.

C_n lies wholly in L_n. If this assertion be false, let k be a piece of the set H which forms part of the boundary of L_n, and contains a point at which C_n leaves L_n. The end points of C_n lie on L_n; hence C_n returns

to L_n. But k divides the whole of M into two regions; hence to return to L_n, C_n would have to return to k, which is impossible, since a reduced curve never returns to a copy of T which it has once left.

We will now prove that if r is any positive integer greater than n, the linear set L_r contains L_n. Let h be any one of the boundary pieces common to two adjacent copies of T of L_n. C_n contains points, not on h, in each of the two regions into which M is divided by h, for otherwise one of the two linear sets into which L_n is divided by h, would contain all the points of C_n and be a linear set containing fewer copies of T than does L_n, contrary to the hypothesis concerning L_n. Hence any connected region that contains all the points of C_n, must contain the two copies of T of L_n that are adjoined along h. Hence any connected region that contains all of the points of C_n, must contain each copy of T of L_n. In particular, L_r contains all the points of C_n, and hence contains L_n.

The set of all copies of T of all regions L_n, for all positive integers n, will accordingly constitute a linear set, which we denote by L. Any point of the given curve is contained on every curve C_n for sufficiently large values of n, and hence is contained in L. We will now prove that L is an unending linear set.

As seen earlier in the proof, the two linear sets into which L_n and hence L is divided by any boundary piece h, common to copies of T of L, each contain points of the given reduced curve that are not on h. Since a reduced curve never returns to a copy of T which it has once left, the removal of h divides the given reduced curve into just two continuous portions, which lie respectively in the two linear sets into which L is divided by h. But each of these continuous portions of the given reduced curve will contain an infinite number of different pieces of the set H, so that each of the two linear sets into which L is divided by h, must contain an infinite number of different copies of T. We conclude that L is an unending linear set.

Any second linear set that contains L, would have to contain each of the linear sets L_n, hence every copy of T of L. Since there is but one linear set joining any two copies of T of M, any linear set containing all the copies of T of L, is identical with L.

The two results of this section may be combined in the following fundamental theorem:

THEOREM I. *There is a one to one correspondence between the set of all unending reduced curves, and the set of all unending linear sets of M, in which each reduced curve corresponds to that linear set in which it is contained.*

PART II.

Some General Properties of Geodesics on M.

§ 14. Hadamard proved that there is on S one and only one geodesic joining two given points, and deformable into an arbitrary curve joining the two given points, and that this geodesic is shorter than any other rectifiable curve deformable into the given curve joining the two given points. With the aid of the results of sections 9 and 10, Hadamard's result becomes the following theorem.

THEOREM II. *There exists on M one and only one geodesic joining two given points, and this geodesic is shorter than any other rectifiable curve joining the two given points.*

COROLLARY I. *On M two geodesics can intersect in but one point.*

COROLLARY II. *On M a geodesic can have no multiple points.*

§ 15. On a surface representable in the manner in which the given surface is representable, there exists one and only one geodesic through a given point, and tangent to a given direction.

DEFINITION. A point on the surface, and a direction tangent to the surface will be called an *element*, and will be said to *define that sensed geodesic* that passes through the initial point of the given element, and is such that its positive tangent direction at that point agrees with the direction of the given element.

If u and v are parameters in any representation of a part of the surface, and if θ' is the angle which a given tangent direction makes at the point $(u'v')$ with the positive tangent to the curve $u = u'$, then $(u'v'\theta')$ will represent an element of the given surface. We shall understand by each statement of metric relations between elements, the same statement of metric relations between the points in space of three dimensions, obtained by considering the complex $(u'v'\theta')$ as the Cartesian coördinates of a point.

§ 16. Let G be any geodesic segment lying on the original uncut surface. G is an extremal in the Calculus of Variations problem of minimizing the arc length, from which theory we can readily obtain the following theorem that describes the nature of the variation of G with variation of its initial element.*

Corresponding to any positive constants, e and h, there exists a positive constant d, so small, that if any two elements, with initial points on the bounded surface S, lie within d of each other, and if a second pair of elements lie respectively on the two geodesics defined by the first two elements, and if further the initial points of this second pair of elements lie respectively at a distance, measured along the given geodesics from the geodesics' initial

* Cf. Bolza, *Vorlesungen über Variationsrechnung*, 1909, p. 219.

points, that is the same in both cases and that does not exceed h, the second pair of elements will lie within e of each other.

§ 17. According to Theorem II, § 14, any given geodesic on M is shorter than any other rectifiable curve joining its end points. According to the theory of the Calculus of Variations, this is a case where there exists on the given geodesic on M no point conjugate to a given point on the given geodesic. A particular consequence* of this result, as given in the theory of the Calculus of Variations, is that, if we vary the end points of a given geodesic segment, there exists further geodesic segments joining these end points and varying continuously with the end points, both in position and in length. We have seen that there is no more than one geodesic segment joining any two points on M. Whence we may say that the length of the geodesic segment, joining two points on the surface M, is a single-valued continuous function of the position of these points. In particular we have the result:

There exists a finite upper limit to the lengths of all geodesic segments joining pairs of points lying on a closed set of points on M.

§ 18. The following statement, of importance in the developments that are to follow, may readily be proved as a consequence of the development given by Hadamard.

Corresponding to two arbitrary positive constants, e and d, there exists a positive constant h, so large, that if on M each point of a first geodesic segment G, of length h, lies within a geodesic distance d of some point of a second geodesic segment, then this second geodesic segment has at least one element within an e of that element of G which lies at the mid point of G.

§ 19. *The Network H Replaced by a Network H′ of Geodesic Segments.* We denote by H', the set of all geodesic segments on M each of which joins two end points of a piece of the set H, defined in § 12.

We shall prove the following lemma:

Two pieces of the set H′ arising from two different pieces of H, can have no points in common other than one end point.

Let P and Q, P' and Q', be respectively the end points of two different pieces of the set H. Denote by h the geodesic segment that joins P to Q, and by h' the geodesic segment that joins P' to Q'. All cases are included in the three following cases:

CASE I. The two given pieces of the set H have an end point in common.

In particular suppose $P = P'$; if $P = P'$, it follows from the nature of the construction of the surface M, that $Q \neq Q'$. Hence, if $P = P'$, the geodesic segments, h and h', have P as a common end point, and are not identical, since $Q \neq Q'$. With the aid of Corollary II, § 14, we conclude that h and h' can have no other point than P in common.

* Cf. Bolza, loc. cit., p. 307.

CASE II. $P \neq P'$, and $Q \neq Q'$, and one or both of the geodesic segments, h and h', form a piece of the geodesic boundary of the surface M.

h and h' are not identical, since $P \neq P'$, and $Q \neq Q'$. In case h were a piece of the geodesic boundary of M, h' could not meet h in other than an end point without passing off from the surface at that point, contrary to the result of Theorem II, § 14.

CASE III. $P \neq P'$, and $Q \neq Q'$, while neither h nor h' form a piece of the geodesic boundary of the surface M.

According to the result of § 7, there exists on M, a simply connected region, bounded by a simple closed curve, and containing the two given pieces of the set H and the geodesic segments h and h'. This region, which we denote by R, we will suppose taken so large that no points of the two given pieces of the set H, nor of the geodesic segments h and h', are boundary points of R unless they are boundary points of M.

$P, Q, P',$ *and* Q', all lie on the boundary of M, and hence on the boundary of R. Since $P \neq P'$, and $Q \neq Q'$, the two given pieces of the set H have no points in common; it follows that P, Q, P', and Q', lie on the boundary of R in the circular order named.

On the other hand, it is an hypothesis of this case, that neither h nor h' are a piece of the geodesic boundary of M; it follows that neither h nor h' have points other than their end points on the boundary of M, and hence of R. A particular consequence is that h divides R into two regions. If now h' crossed h, its end point P' would lie in one of the two regions into which R is divided by h, while its remaining end point, Q', would lie in the other such region. P, P', Q, Q', would thus lie on the boundary of R in the circular order named. From this contradiction we infer the truth of the lemma in this case. The proof is thus given in general.

§ 20. We seek now to render the original surface simply connected by means of cuts made along geodesics.

In accordance with the result given in § 14, the end points of each one of the cuts by means of which the original surface was rendered simply connected, can be joined on the original surface by a geodesic segment deformable into the given cut. With the aid of the result of § 9, it appears that these geodesic segments correspond on M to the geodesic segments of H', and further that two of these geodesic segments arising from different cuts correspond on M only to different members of the set H'. From the result of the preceding section we conclude that no two of these geodesic segments have any points in common other than their end points.

If now the original surface be cut along these geodesic segments, the resulting surface, which we denote by U, will be simply connected. The infinitely leaved surface M can be formed by joining together copies of U in the same manner as it was formed from copies of T.

We define a reduced curve of the set H', and a linear set of copies of U, by replacing H and T in the definitions of sections 11 and 12, by H' and U, respectively. In the same manner, we obtain from the results of sections 11 and 13, the following results in terms of H' and U.

(*A*) *If U' and U'' are any two copies of U of the copies of U that make up M, there exists one and only one linear set of which U' is the first and U'' the last member.*

(*B*) *There is a one to one correspondence between the set of all unending reduced curves of the set H', and the set of all unending linear sets of copies of U, in which each reduced curve corresponds to that linear set in which it is contained.*

Geodesics Lying Wholly on M.

§ 21. Let G be a geodesic lying wholly on M. G cannot become infinite in length in any one copy of U, as follows from the result of section 17. In leaving a copy of U of M, G cannot be tangent to any one of the geodesic segments that separate that copy of U from the remainder of M. For in such a case G would coincide with that geodesic segment, and pass off from M at its end points. Further, it follows from Corollary I, section 14, that G can have but one point of intersection with any of the geodesic segments that separate the different copies of U of M. Hence G never returns to a copy of U of M which it has once left. Since G cannot become infinite in length in any one copy of U, it follows that either of the two portions into which G may be divided by an arbitrary one of its points, has points in common with an infinite number of different copies of U.

From these last two results, it follows by a proof, which except for terminology, may be given as a repetition of the proof of (*B*), section 13, that *a geodesic lying wholly on M, is contained in one and only one linear set, which set must be an unending linear set.*

§ 22. As a converse to the preceding result, we have the following:

If there be given any unending linear set, there exists one, and only one geodesic contained wholly in the given linear set, and this geodesic has at least one point in each copy of U of the given linear set.

Let an unending linear set of copies of U be given as follows:

$$\cdots, U_{-2}, U_{-1}, U_0, U_1, U_2, \cdots.$$

Let g_n be a geodesic segment joining any interior point of U_{-n} to an interior point of U_n. The set of copies of U which g_n passes through, is seen to form a linear set joining U_{-n} to U_n. But from the result of section 11, it follows that there is but one such linear set joining U_{-n} to U_n. The linear set,

$$U_{-n}, U_{-n+1}, \cdots, U_0, \cdots, U_{n-1}U_n,$$

is one such set; we conclude that this is the set of copies of U through which g_n passes.

Denote by E_n an element on that part of g_n that lies in U_0. The set of all elements E_n will have a limit element. Let G be the geodesic defined by this element. We will first show that G has a point in each copy of U of the given linear set.

Let r be any positive integer. For integers $n > r$ that portion of g_n in

$$U_{-r}, U_{-r+1}, \cdots, U_0, \cdots, U_{r-1}U_r, \tag{1}$$

is according to the result of section 17, less in length than some fixed quantity independent of n. Now a finite segment of a geodesic varies continuously with its initial element. It follows that G possesses a finite segment, say G_r, which has a point in each copy of U of (1), and which is wholly contained in the set (1). From the fact that G_r has a point in each copy of U of (1), we may conclude that G has a point in each copy of the given linear set.

We seek to prove that G is wholly contained in the given linear set. Any finite segment of any geodesic on M has points in not more than a finite number of copies of U. Cf. section 7. We may conclude that, for r sufficiently large, any given segment of G that begins with a point of U_0, is included in one of the two portions into which G_r is divided by that point. Thus every point of G lies on some segment G_r. But every point of G_r, and hence every point of G lies in the given linear set.

If there were a second geodesic, say G', contained in the same linear set as G, it would follow from the result of section 18, that every element of G would be a limit element of elements on G'. This is impossible, since G' can never return to a copy of U which it has once left.

The results of this section and the preceding, are summed up in the following:

THEOREM III. *There is a one-to-one correspondence between the set of all geodesics lying wholly on M, and the set of all unending linear sets, in which each geodesic corresponds to that linear set in which it is contained.*

A similar theorem is now obtained from (B) in § 20.

THEOREM IV. *There is a one-to-one correspondence between the set of all geodesics lying wholly on M, and the set of all reduced curves of the set H', in which each geodesic corresponds to that reduced curve that is contained in the same linear set.*

Congruent Curves.

§ 23. DEFINITION. Two curves on M that correspond to the same curve on S will be said to be *congruent.*

The first statement of the following theorem serves as an existence proof for closed geodesics on the original surface.

(*A*) *If a reduced curve on M consists of successive mutually congruent portions, the geodesic that lies in the same linear set as the given reduced curve, consists also of successive mutually congruent portions. Conversely, if a geodesic lying wholly on M consists of successive mutually congruent portions, the reduced curve that lies in the same linear set, consists also of successive mutually congruent portions.*

Suppose the given reduced curve consists of successive mutually congruent portions. Let the configuration, consisting of the given reduced curve and the geodesic lying in the same linear set, be carried as a whole into that congruent configuration in which one of the successive mutually congruent portions of the given reduced curve is carried into the succeeding congruent portion. Denote this transformation by T.

The given geodesic is carried into itself by T. For otherwise there would exist two different geodesics, namely the given geodesic and the geodesic into which it is carried by T, both lying in the same linear set as the given reduced curve. This is contrary to the result of Theorem IV, section 22.

Now let P' be any point on the given geodesic. Let P'' be the point in which P' is carried by T. P'' lies on the given geodesic, since the given geodesic is carried into itself by T. The successive images of the geodesic segment $P'P''$ which result through repetition of T and its inverse, will be successive mutually congruent portions of the given geodesic, all of whose points can be reached by a sufficient number of such transformations.

The converse statement of the theorem can be proved in a similar manner.

In the same manner also, the following can be established.

(*B*) *Let there be given a first reduced curve and geodesic contained in a single linear set, and a second reduced curve and geodesic that are also contained in a single linear set. If the two given reduced curves are congruent, the two geodesics must also be congruent; and conversely, if the two given geodesics are congruent, the two reduced curves must also be congruent.*

Variation of Geodesics with Their Initial Elements.

§ 24. The following statement depends upon the result given in section 18.

(*A*) Corresponding to any positive quantity e, there exists a positive integer n, so large, that if any two unending linear sets have in common a region U, say U', together with the first n regions U succeeding U' in either sense in the given linear set, then there exists in U', and on each of the two geodesics that pass respectively through the two given linear sets, at least one element that lies within e of some element on that part of the other geodesic that lies in U'.

The converse statement depends upon the property of continuous variation of a geodesic with its initial element, as given in section 16.

(*B*) Corresponding to any positive integer n, there exists a positive constant e, so small, that if on each of two geodesics there exists some element within e of some element on the other, then the two linear sets through which the two given geodesics respectively pass, possess in common that linear set that contains the region or regions U, in which the two given elements lie, together with n regions U succeeding these regions in either sense.

By virtue of the relations between unending reduced curves and corresponding unending linear sets, as given in the proofs of section 13, the statements of (*A*) and (*B*) of this section become the following:

THEOREM VI. *Corresponding to any positive constant e, there exists a positive constant k, so large, that if two unending reduced curves possess in common a continuous segment of length exceeding k, the two corresponding geodesics each have at least one element within e of some element on the other, and with initial point in the same copy of U as the mid point of the common reduced segment. Conversely, corresponding to any positive constant k, there exists a positive constant d, so small, that if on each of two geodesics there exists some element within d of some element on the other, the two corresponding reduced curves possess in common a segment of length k, with mid point in the same copy of U as the initial point of either of the two elements.*

§ 25. DEFINITION. Two elements E' and E'' of a set of elements E on M, will be said to be *mutually accessible* in E, if corresponding to any positive constant e, there exists in the set E a finite ordered subset of elements of which the first element is E', and the last element E'', while each element of the subset, excepting the last, lies within e of the following element.

THEOREM VI. *In the set of all elements on a closed region of M, and on geodesics lying wholly on M, no two elements lying on different geodesics are mutually accessible.*

Let G' and G'' be two different geodesics lying wholly on M. Let E' and E'' be two elements lying respectively on G' and G''. Let R be any closed region of M, in which the initial points of E' and E'' lie. We will show that E' and E'' are not mutually accessible in the set of all elements on R, and on geodesics lying wholly on M.

G' and G'' are different geodesics, and are accordingly not contained in the same linear set of copies of U. There therefore exists some geodesic segment, of the geodesic segments that separate different copies of U of M, that G' crosses and that G'' does not cross. Denote this geodesic segment by h. Denote by B' the set of all elements on R, and on geodesics that lie

wholly on M and cross h. Denote by B'' the set of all elements on R, and that lie on geodesics that lie wholly on M and do not cross h.

B' is a closed set. For it follows from the result of sections 16 and 17, that a limit element of elements of the set B', defines a geodesic, say G, lying wholly on M, and that is either tangent to h or else crosses h. But G cannot be tangent to h without passing off from M at the end points of h. Hence G crosses h. Thus B' is a closed set. In a similar manner it follows that B'' is a closed set.

From their definitions it follows that B' and B'' can have no elements in common. There accordingly exists a positive constant d, such that no element of B' lies within d of any element of B''. Hence E', which belongs to B', and E'', which belongs to B'', cannot be mutually accessible in the set of elements comprising the elements of B' and B''. But the elements of B' and B'' comprise all the elements on geodesics lying wholly on M, and with initial points in the region R.

The theorem thus is proved.

HARVARD UNIVERSITY,
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CONJUGATE SYSTEMS WITH INDETERMINATE AXIS CURVES.

By Ernest P. Lane.

I. Introduction.

The general projective theory of congruences may be studied* in connection with a completely integrable system of partial differential equations of the form

$$D) \qquad \begin{aligned} y_v &= mz, \qquad z_u = ny, \\ y_{uu} &= a\,y + b\,z + c\,y_u + d\,z_v, \\ z_{vv} &= a'y + b'z + c'y_u + d'z_v, \end{aligned}$$

where the subscripts indicate differentiations. In order that system (D) may be completely integrable, its coefficients, which are assumed to be analytic functions of the two arguments u and v, must satisfy the following integrability conditions:

$$c = f_u, \quad d' = f_v, \quad b = -d_v - df_v, \quad a' = -c_u - c'f_u, \quad mn - c'd = f_{uv},$$

$$m_{uu} + d_{vv} + df_{vv} + d_v f_v - f_u m_u = ma + db',$$

$$n_{vv} + c'_{uu} + c'f_{uu} + c'_u f_u - f_v n_v = nb' + c'a,$$

$$2m_u n + mn_u = a_v + f_u mn + a'd,$$

$$m_v n + 2mn_v = b'_u + f_v mn + bc',$$

where f is an analytic function of u and v. Then system (D) has exactly four pairs of linearly independent solutions, $(y^{(i)}, z^{(i)})$, $(i = 1, \cdots 4)$, such that the general solution is of the form

$$y = \sum_{i=1}^{4} c^{(i)} y^{(i)}, \qquad z = \sum_{i=1}^{4} c^{(i)} z^{(i)},$$

where $c^{(1)}, \cdots, c^{(4)}$ are arbitrary constants. If $y^{(1)}, \cdots, y^{(4)}$ and $z^{(1)}, \cdots, z^{(4)}$ are interpreted as the homogeneous coördinates of two points P_y and P_z of three-dimensional space, these points will describe in general, for variable u and v, two surfaces S_y and S_z, but the possibility is not excluded that these surfaces reduce to curves. The line $P_y P_z$ describes a congruence whose

* E. J. Wilczynski, "Sur la théorie générale des congruences." Mémoires publies par la classe des Sciences de l'Academie Royale de Belgique. Collection en 4°. Deuxième série. Tome III (1911). This paper will be cited hereafter as Brussels Paper.

focal surfaces are S_y and S_z, and whose developables are obtained by equating either u or v to a constant.

The form of system (D) is undisturbed by the group of transformations

$$(T) \qquad y = \lambda(u)\bar{y}, \qquad z = \mu(v)\bar{z}, \qquad \bar{u} = \alpha(u), \qquad \bar{v} = \beta(v),$$

where $\lambda, \mu, \alpha, \beta$ are arbitrary functions of the arguments indicated. Among the invariants of this group are the four coefficients m, n, c', d. It is easy to verify that I and l are also invariants, where

$$I = a + c\frac{m_u}{m} - \frac{m_{uu}}{m}, \qquad l = \frac{b}{d} + \frac{n_v}{n}.$$

Five fundamental covariants are $y, z, \rho, \sigma, \epsilon$, where

$$\rho = y_u - \frac{m_u}{m}y, \qquad \sigma = z_v - \frac{n_v}{n}z, \qquad \epsilon = \rho_u - \frac{m_{1u}}{m_1}\rho,$$

and

$$m_1 = m\left(mn - \frac{\partial^2}{\partial u \partial v}\log m\right).$$

The axis of a point P on a surface has been defined,* with reference to a given conjugate system of curves on the surface, to be the line of intersection of the osculating planes at P of the two curves of the system which pass through P. The axis congruence of a surface is composed of the axes of all its points. An axis curve of a surface is a curve such that the axes of its points form a developable of the axis congruence.

The curves $u =$ const. and $v =$ const. form a conjugate system on S_y. For convenience of reference we quote here some results,† concerning the axis curves of S_y defined with reference to this conjugate system. The axis curves are given by the equation

$$(1) \qquad d_1\delta u^2 - mI\delta u\delta v - c'dm\delta v^2 = 0,$$

where

$$d_1 = d\left(c'd - \frac{\partial^2}{\partial u \partial v}\log d\right).$$

If the function τ be defined by

$$(2) \qquad \tau = lz + \sigma,$$

then the point P_τ is the point of intersection of the axis of P_y with the line P_zP_σ. The foci of the axis are determined by factoring the covariant

$$(C) \qquad c'd_1y^2 - mIy\tau - dm\tau^2.$$

* E. J. Wilczynski, "The General Theory of Congruences," *Transactions of the American Mathematical Society*, Vol. XVI, No. 3, pp. 311–327. This paper will be cited hereafter as Congruences.

† Congruences, § 2.

It has been remarked* that the axis curves are indeterminate if

$$d_1 = mI = c'dm = 0. \tag{3}$$

It is the purpose of this paper to pursue the consequences of this remark, and to study those surfaces upon which there exists a conjugate system with indeterminate axis curves.

The author takes advantage of this opportunity to express his appreciation of the many helpful suggestions so kindly given him by Professor Wilczynski while this paper was in process of preparation.

I. FUNDAMENTAL THEOREM.

We shall now assume that the axis curves on S_y defined with reference to the parametric conjugate system, are indeterminate. Then the equations

$$d_1 = mI = c'dm = 0 \tag{3 bis}$$

are valid. If the surface S_y does not reduce to a curve, the invariant m does not vanish.† Similarly, if S_z is not degenerate, $n \neq 0$. Moreover, if S_y is not a developable surface, $d \neq 0$. We shall assume from now on that we are considering only those surfaces S_y which are nondegenerate and nondevelopable, and surfaces S_z which at least are nondegenerate. Under these assumptions we shall proceed to find formulas for the coefficients of system (D).

Equations (3) may be replaced by the equivalent equations

$$\frac{\partial^2}{\partial u \partial v} \log d = I = c' = 0. \tag{4}$$

As an immediate consequence we have

$$d = UV, \tag{5}$$

where U and V are nonvanishing functions of u alone and of v alone, repsectively.

When use is made of equations (4) and (5), the integrability conditions may be written in the form

$$c = f_u, \quad d' = f_v, \quad b = -UV' - UVf_v, \quad a' = 0,$$

$$mn = f_{uv}, \quad V'' + Vf_{vv} + V'f_v = Vb', \quad n_{vv} - f_v n_v = nb',$$

$$2m_u n + mn_u = a_v + f_u mn,$$

$$m_v n + 2mn_v = b'_u + f_v mn,$$

* Congruences, p. 317.

† Brussels Paper, p. 28.

where V', V'', etc., denote derivatives. On eliminating b' between the sixth and ninth conditions, on making use of the fifth condition, and on integrating with respect to v there results

$$n = U_1 V e^f, \tag{6}$$

where U_1 is an arbitrary nonvanishing function of u alone. By means of (6) and the fifth and eighth conditions we find, after integrating with respect to v,

$$a = 2f_{uu} - f_u \frac{U_1'}{U_1} - f_u^2 - U_2,$$

where U_2 is still another arbitrary function of u only.

Let us now regard the functions f, V, U, U_1, U_2 as given functions, and let us consider the possibility of defining the coefficients of a system (D) by the following formulas:

$$\begin{aligned}
m &= \frac{f_{uv} e^{-f}}{U_1 V}, & n &= U_1 V e^f. \\
a &= 2f_{uu} - f_u \frac{U_1'}{U_1} - f_u^2 - U_2, & b &= -UV' - UVf_v, \\
c &= f_u, & d &= UV, \\
a' &= 0, & b' &= \frac{V''}{V} + f_{vv} + \frac{V'}{V} f_v, \\
c' &= 0, & d' &= f_v.
\end{aligned} \tag{7}$$

When the coefficients thus defined are substituted in the integrability conditions, all of them are found to be satisfied identically except the sixth. This condition is found to be equivalent to $I = 0$. When it is written out in terms of the given functions, it becomes

$$\begin{aligned}
&\frac{f_{uuuv}}{f_{uv}} - \frac{3f_u f_{uuv}}{f_{uv}} - 3f_{uu} + 3f_u^2 - \frac{2U_1' f_{uuv}}{U_1 f_{uv}} \\
&\qquad + 4f_u \frac{U_1'}{U_1} + 2\left(\frac{U_1'}{U_1}\right)^2 - \frac{U_1''}{U_1} + U_2 = 0.
\end{aligned} \tag{8}$$

Therefore if the functions f, U_1, U_2 satisfy equation (8), formulas (7) furnish the coefficients of a system (D) which satisfies all of the requirements.

Our results may be summarized in the following fundamental theorem. *Let U, U_1, and U_2 be arbitrary functions of u alone such that $UU_1 \neq 0$. Furthermore let f be a function of u and v satisfying equation* (8) *and such that $f_{uv} \neq 0$. Finally let V be an arbitrary nonvanishing function of v alone. Then formulas* (7) *furnish the coefficients of the most general system of the form*

(D) *for which S_y is nondegenerate and nondevelopable and S_z nondegenerate, and for which the axis curves on S_y, defined with reference to the parametric conjugate system, are indeterminate.* The integrability conditions for such a system (D) are all satisfied identically.

III. A Simplifying Transformation.

All transformations of the group (T) are without geometrical significance; we shall therefore employ this group to simply our analysis. The effect of (T) on (D) is to transform (D) into another system of the same form, of which the coefficients have the following values.*

$$
\begin{aligned}
&\bar{m} = \frac{\mu}{\lambda}\frac{1}{\beta_v}m, && \bar{n} = \frac{\lambda}{\mu}\frac{1}{\alpha_u}n,\\
&\bar{a} = \frac{1}{\alpha_u^2}\left(a + \frac{\lambda_u}{\lambda}c - \frac{\lambda_{uu}}{\lambda}\right), && \bar{b}' = \frac{1}{\beta_v^2}\left(b' + \frac{\mu_v}{\mu}d' - \frac{\mu_{vv}}{\mu}\right),\\
(9)\qquad &\bar{b} = \frac{1}{\alpha_u^2}\frac{\mu}{\lambda}\left(b + \frac{\mu_v}{\mu}d\right), && \bar{a}' = \frac{1}{\beta_v^2}\frac{\lambda}{\mu}\left(a' + \frac{\lambda_u}{\lambda}c'\right),\\
&\bar{c} = \frac{1}{\alpha_u}\left(c - 2\frac{\lambda_u}{\lambda} - \frac{\alpha_{uu}}{\alpha_u}\right), && \bar{d}' = \frac{1}{\beta_v}\left(d' - 2\frac{\mu_v}{\mu} - \frac{\beta_{vv}}{\beta_v}\right),\\
&\bar{d} = \frac{\beta_v}{\alpha_u^2}\frac{\mu}{\lambda}d, && \bar{c}' = \frac{\alpha_u}{\beta_v^2}\frac{\lambda}{\mu}c'.
\end{aligned}
$$

In the first place we observe that the product UV is reduced to unity by a transformation of the group (T) for which

$$\lambda = U, \qquad \mu V = \alpha_u = \beta_v = 1.$$

When this transformation has been made the coefficient $\bar{d}$ of the transformed system beocmes equal to unity, and the integrability conditions reduce to

$$
\begin{aligned}
c = f_u, \qquad d' = f_v, \qquad b = -f_v, \qquad a' = 0,\\
mn = f_{uv}, \qquad b' = f_{vv}, \qquad n_{vv} = f_v n_v + nb',\\
2m_u n + mn_u = a_v + mnf_u,\\
m_v n + 2mn_v = b'_u + mnf_v.
\end{aligned}
$$

On eliminating b' between the sixth and ninth conditions, on making use of the fifth condition, and on integrating with respect to v, there results

$$n = U_1 e^f.$$

The largest subgroup of the group (T) which preserves the condition $d = 1$ is given by the relations

$$\alpha_u^2\lambda = \beta_v\mu = \text{const.},$$

* Brussels Paper, Equations (16) and (22).

where λ and μ are still arbitrary functions. The effect of this subgroup on the coefficient n is given by

$$\bar{n} = \frac{\beta_v}{\alpha_u^3} n = \frac{\beta_v}{\alpha_u^3} U_1 e^f.$$

Therefore the function U_1 is reduced to unity by a transformation for which

$$\alpha_u^2 \lambda = \mu = 1, \qquad \alpha_u^3 = U_1, \qquad \beta_v = 1.$$

When this transformation has been made, the fifth integrability condition gives

$$m = f_{uv} e^{-f},$$

and then the relation $I = 0$ becomes

$$a = \frac{f_{uuuv}}{f_{uv}} - \frac{3 f_u f_{uuv}}{f_{uv}} - f_{uu} + 2 f_u^2. \tag{10}$$

Now regarding f as a given function, we consider the possibility of defining the coefficients of a system (D) by the following formulas:

$$\begin{aligned} m &= f_{uv} e^{-f}, & n &= e^f, \\ a &= \frac{f_{uuuv}}{f_{uv}} - \frac{3 f_u f_{uuv}}{f_{uv}} - f_{uu} + 2 f_u^2, & b &= -f_v, \\ c &= f_u, & d &= 1, \\ a' &= 0, & b' &= f_{vv}, \\ c' &= 0, & d' &= f_v. \end{aligned} \tag{11}$$

When the coefficients thus defined are substituted in the integrability conditions, all of them are found to be satisfied identically except the eighth, which becomes

$$\frac{f_{uuuvv}}{f_{uv}} - \frac{3 f_u f_{uuvv}}{f_{uv}} - \frac{f_{uuuv} f_{uvv}}{(f_{uv})^2} + \frac{3 f_u f_{uuv} f_{uvv}}{(f_{uv})^2} - 6 f_{uuv} + 6 f_u f_{uv} = 0. \tag{12}$$

Therefore if f satisfies equation (12), then formulas (11) furnish the coefficients of a system (D) such that the axis curves of the parametric conjugate system are indeterminate.

The apparently formidable equation (12) can be very much simplified. On rearranging the terms and integrating with respect to v, we obtain

$$\frac{f_{uuuv}}{f_{uv}} - \frac{3 f_u f_{uuv}}{f_{uv}} - 3 f_{uu} + 3 f_u^2 + U_2 = 0, \tag{13}$$

where U_2 is an arbitrary function of u alone. If equation (13) be multiplied through by f_{uv}, it can be integrated with respect to v, giving

$$f_{uuu} - 3 f_u f_{uu} + f_u^3 + U_2 f_u = U_3, \tag{14}$$

where U_3 is another arbitrary function of u alone. Finally we introduce the function F by the definition

$$f = -\log F, \tag{15}$$

whereupon (14) reduces to the linear homogeneous equation

$$\frac{\partial^3 F}{\partial u^3} + U_2 \frac{\partial F}{\partial u} + U_3 F = 0. \tag{16}$$

We remark that as a consequence of (15) we have

$$F^2 f_{uv} + F F_{uv} - F_u F_v = 0.$$

We may now state the following theorem. *Let U_2 and U_3 be arbitrary functions of the single variable u. Let F be a function of u and v which is a solution of* (16) *but which is not a solution of $FF_{uv} - F_u F_v = 0$. Let the function f be determined from $f = -\log F$. Then the most general system (D) which is of interest in the theory of indeterminate axis curves can be reduced, by means of transformations of group (T), to the system:*

$$(\Delta) \qquad \begin{aligned} y_v &= f_{uv} e^{-f} z, \qquad z_u = e^f y, \\ y_{uu} &= \left(\frac{f_{uuuv}}{f_{uv}} - \frac{3 f_u f_{uuv}}{f_{uv}} - f_{uu} + 2 f_u^2 \right) y - f_v z + f_u y_u + z_v, \\ z_{vv} &= f_{vv} z + f_v z_v. \end{aligned}$$

The integrability conditions for system (Δ) are all satisfied identically. We shall base our further theory on this system.

IV. The First Laplace Transformed Congruence.

In the theory of indeterminate axis curves, the first Laplace transformed congruence, whose focal surfaces are S_ρ and S_y, and the original congruence, whose focal surfaces are S_y and S_z and whose equations are system (D), are equally important. We shall need to set up the system (D_1) of equations of the first Laplace transformed congruence. But before doing so let us define a new function g by the equation

$$g = 2f - \log f_{uv}. \tag{17}$$

As thus defined, g plays the same rôle in the theory of the first Laplace transformed congruence that f plays in the theory of the original congruence.

By means of equation (17) it is possible to write (13) in the symmetrical form

$$f_u^2 + g_u^2 - f_{uu} - g_{uu} - f_u g_u + U_2 = 0. \tag{18}$$

Differentiation of (18) with respect to v yields

$$2 f_u f_{uv} + 2 g_u g_{uv} - f_{uuv} - g_{uuv} - f_{uv} g_u - f_u g_{uv} = 0. \tag{19}$$

Now the sum of the first, third, and fifth terms of this equation is easily shown to be zero, by using equation (17). Therefore, if we suppose $g_{uv} \neq 0$, an assumption whose geometrical meaning will be made clear later, we have

$$f_u = 2g_u - \frac{g_{uuv}}{g_{uv}}. \tag{20}$$

We obtain, on integrating with respect to u,

$$f = 2g - \log g_{uv} + \log V, \tag{21}$$

where V is an arbitrary function of v alone.

We may make a transformation to reduce V to unity. It is readily verified that all transformations of the group (T) for which

$$\lambda = \alpha_u, \qquad \alpha_u^2 \lambda = \beta_v \mu = 1, \tag{22}$$

leave the form of system (Δ) unchanged, although f is thereby transformed according to the formula

$$\bar{f} = f - \log \mu. \tag{23}$$

Moreover, g is transformed according to the formula

$$\bar{g} = g - 3 \log \mu, \tag{24}$$

as is seen upon using equation (17). If then we choose

$$\lambda = 1, \qquad \mu = V^{-\frac{1}{3}}, \qquad \alpha_u = 1, \qquad \beta_v = V^{\frac{1}{3}}, \tag{25}$$

the conditions (22) are satisfied and we find, on making the necessary calculations, that

$$\bar{f} = 2\bar{g} - \log \bar{g}_{\bar{u}\bar{v}},$$

so that the transformed $\bar{V}$ is unity. Equation (17) may now be solved for f in the form

$$f = 2g - \log g_{uv}. \tag{26}$$

When f is eliminated from equation (14), the result is

$$g_{uuu} - 3g_u g_{uu} + g_u^3 + U_2 g_u - U_2' + U_3 = 0. \tag{27}$$

Let us introduce the function G by the definition

$$g = -\log G. \tag{28}$$

Then (27) is equivalent to

$$\frac{\partial^3 G}{\partial u^3} + U_2 \frac{\partial G}{\partial u} + (U_2' - U_3) G = 0. \tag{29}$$

It will be observed that this equation is the Lagrange adjoint of equation (16), which is satisfied by F, the derivatives in both equations being taken with respect to the variable u only.

It is interesting at this point to see that F and G satisfy also certain equations in which differentiations occur only with respect to the variable v. The truth of the equation

$$2f_v f_{uv} + 2g_v g_{uv} - f_{vvu} - g_{vvu} - f_{uv}g_v - g_{uv}f_v = 0$$

is established by observing that alternate terms taken together form two sums, one of which vanishes in virtue of equation (17), and the other in virtue of (26). Upon integrating with respect to u, we obtain

$$f_v^2 + g_v^2 - f_{vv} - g_{vv} - f_v g_v + V_2 = 0,$$

where V_2 is an arbitrary function of v alone. When f is eliminated from this equation, there results

$$\frac{g_{vvvu}}{g_{uv}} - \frac{3g_v g_{vvu}}{g_{uv}} - 3g_{vv} + 3g_v^2 + V_2 = 0.$$

Upon integrating with respect to u again, we obtain

$$g_{vvv} - 3g_v g_{vv} + g_v^3 + V_2 g_v = V_3, \tag{30}$$

where V_3 is a function of v only. Equation (30) is equivalent to

$$\frac{\partial^3 G}{\partial v^3} + V_2 \frac{\partial G}{\partial v} + V_3 G = 0. \tag{31}$$

Similarly, if g is eliminated from (30), there results

$$f_{vvv} - 3f_v f_{vv} + f_v^3 + V_2 f_v - V_2' + V_3 = 0,$$

and this equation is equivalent to

$$\frac{\partial^3 F}{\partial v^3} + V_2 \frac{\partial F}{\partial v} + (V_2' - V_3)F = 0. \tag{32}$$

Equation (32) is the Lagrange adjoint of (31).

Making use of the function g, let us place*

$$\rho = y_u - (f_u - g_u)y, \qquad \zeta = e^{g-f}y, \tag{33}$$

and find the system (Δ_1) of equations of the first Laplace transformed congruence. On differentiating (33) and employing system (Δ), we find that ρ and ζ satisfy

$$\begin{aligned} &\rho_v = e^g \zeta, \qquad \zeta_u = e^{g-f}\rho, \\ (\Delta_1) \qquad &\rho_{uu} = g_{uu}\rho + g_u \rho_u, \\ &\zeta_{vv} = -g_u \rho + (g_{vv} - f_{vv} - f_v g_v + f_v^2)\zeta + \rho_u + g_v \zeta_v. \end{aligned}$$

* Brussels Paper, pp. 66–67.

V. GEOMETRICAL THEOREMS.

The three covariants ρ, σ, ϵ for system (Δ), when written in terms of f and g, are

$$\rho = y_u - (f_u - g_u)y, \qquad \sigma = z_v - f_v z, \qquad \epsilon = \rho_u - g_u\rho. \tag{34}$$

These equations, together with systems (Δ) and (Δ_1) will now be used to deduce a number of geometric propositions.

From the first of equations (34) we find, by differentiation with respect to u and use of system (Δ), that $\epsilon = \sigma$. Moreover we find $\sigma_u = \sigma_v = 0$.

Therefore, *if the axis curves of a conjugate net are indeterminate, the second and minus second Laplace transforms of the net are identical and reduce to a single fixed point.*

The third equation of system (Δ_1) shows that the lines v = const. on S_ρ are straight lines, while the third of equations (34) shows that these lines all pass through P_ϵ. Therefore the surface S_ρ is a cone with its vertex at P_ϵ. In the same way we conclude from the fourth equation of system (Δ) that the lines u = const. on S_z are straight lines, and from the second of equations (34) that these lines all pass through P_σ. Therefore, *if the axis curves of a conjugate net are indeterminate, the first and minus first Laplace transforms of the net are on two cones with a common vertex at the fixed point into which the second and minus second Laplace transforms degenerate.*

Inspection of the covariant (C), namely

$$c'd_1y^2 - mIy\tau - dm\tau^2, \tag{C bis}$$

shows that, when the axis curves on S_y are indeterminate, the foci of the axis of P_y coincide at the point P_τ. Moreover equation (2) shows that P_τ coincides with the point P_σ. Therefore *if the axis curves of a conjugate system on a surface are indeterminate, the axis congruence reduces to a bundle of lines with its vertex at the fixed point into which the second and minus second Laplace transforms of the surface degenerate.*

Let us differentiate twice with respect to v the first equation of system (Δ). From the equations thus obtained let us eliminate z, z_v, and z_{vv}, using the fourth equation of system (Δ). There results

$$y_{vvv} + (2g_v - 3f_v)y_{vv} + (g_{vv} + g_v^2 + 2f_v^2 - 2f_{vv} - 3f_vg_v)y_v = 0, \tag{35}$$

the equation of the curves u = const. on S_y. Since this equation is of the third order we conclude that the curves u = const. are plane curves. Similarly we differentiate once with respect to u the third equation of system (Δ) and eliminate z, z_v and z_{uv} to obtain

$$y_{uuu} - f_uy_{uu} + (g_{uu} - 2f_{uu} + f_ug_u - g_u^2)y_u + (g_{uuu} - f_{uuu} + f_ug_{uu} + g_uf_{uu} - 2g_ug_{uu})y = 0, \tag{36}$$

the equation of the curves $v = \text{const.}$ on S_y. These curves are also plane curves. In this way we see that, *if the axis curves of a conjugate system are indeterminate, this conjugate system consists of two one-parameter families of plane curves.*

VI. Integration of System (Δ).

It has been shown that, for system (Δ), the two surfaces S_σ and S_ϵ reduce to the same fixed point, which may be denoted indifferently by P_σ or P_ϵ. Since the fundamental tetrahedron of reference is arbitrary, let us choose it so that P_σ may have (0, 0, 0, 1) for its coördinates. Then the second of equations (34) frunishes

$$z_v^{(k)} - f_v z^{(k)} = 0, \qquad (k = 1, 2, 3),$$

$$z_v^{(4)} - f_v z^{(4)} = 1.$$

After integrating with respect to v, we obtain

$$z^{(k)} = \varphi^{(k)} e^f,$$

$$z^{(4)} = \varphi^{(4)} e^f + e^f \int e^{-f} dv,$$

where the four functions φ are functions of u alone, and are as yet arbitrary. By the indicated quadrature, we mean the definite integral from a fixed lower limit to a variable upper limit, u being regarded as fixed.

We now use the second equation of system (Δ) to determine four values of y, and obtain

$$y^{(k)} = \varphi^{(k)} f_u + \varphi_u^{(k)},$$

$$y^{(4)} = \varphi^{(4)} f_u + \varphi_u^{(4)} + f_u \int e^{-f} dv - \int e^{-f} f_u dv.$$

When corresponding values of y and z are substituted in the first and fourth equations of system (Δ), these equations are found to be satisfied identically without restriction on the functions φ. But when corresponding values of y and z are substituted in the third equation, it is found that the functions φ must be solutions of certain ordinary differential equations. In fact $(y^{(k)}, z^{(k)})$, $(k = 1, 2, 3)$, *constitute three pairs of linearly independent solutions of system* (Δ) *if, and only if, the* $\varphi^{(k)}$ *are three linearly independent solutions of the equation**

$$\frac{d^3\varphi}{du^3} + U_2 \frac{d\varphi}{du} + U_3 \varphi = 0, \tag{37}$$

* Since the second derivative is missing, the Wronskian of any three linearly independent solutions is a constant, which may be supposed to be unity.

and $(y^{(4)}, z^{(4)})$ *are a pair of solutions of system* (Δ) *if, and only if,* $\varphi^{(4)}$ *is a solution of the equation*

$$\frac{d^3\varphi}{du^3} + U_2\frac{d\varphi}{du} + U_3\varphi = 1. \tag{38}$$

Equations (37) and (16) are to be compared.

Let us denote the three second order Wronskians of the $\varphi^{(k)}$ by $U^{(k)}$, so that, for instance,

$$U^{(1)} = \varphi_u^{(2)}\varphi^{(3)} - \varphi_u^{(3)}\varphi^{(2)}.$$

Then the functions $U^{(k)}$ are solutions of the Lagrange adjoint of (37), namely

$$\frac{d^3U}{du^3} + U_2\frac{dU}{du} + (U_2' - U_3)U = 0. \tag{39}$$

This equation is to be compared with equation (29) which is satisfied by G.

We remark further that comparison of (37) and (38) shows that $\varphi^{(4)}$ can be expressed in terms of the $\varphi^{(k)}$ by the well-known method of *variation of parameters*. We find in this way

$$\varphi^{(4)} = \sum_{k=1}^{3}\left(\int U^{(k)}du + c^{(k)}\right)\varphi^{(k)}, \tag{40}$$

where the $c^{(k)}$ are arbitrary constants, and the indicated quadratures denote again definite integrals from a common fixed lower limit to a variable upper limit.

In the process of integrating system (Δ) we have used the second of equations (34), but we might just as well have used the third. This equation gives

$$\rho_u^{(k)} - g_u\rho^{(k)} = 0, \qquad (k = 1, 2, 3),$$

$$\rho_u^{(4)} - g_u\rho^{(4)} = 1.$$

On integrating with respect to u we obtain

$$\rho^{(k)} = \psi^{(k)}e^{g},$$

$$\rho^{(4)} = \psi^{(4)}e^{g} + e^{g}\int e^{-g}du,$$

where the functions ψ are functions of v alone, and are as yet arbitrary. The first equation of system (Δ_1) gives

$$\zeta^{(k)} = \psi_v^{(k)} + g_v\psi^{(k)},$$

$$\zeta^{(4)} = \psi_v^{(4)} + g_v\psi^{(4)} + g_v\int e^{-g}du - \int e^{-g}g_v du.$$

Then the second of equations (33) furnishes four values for y, and the first of system (Δ) furnishes four values for z.

We find that the $\psi^{(k)}$ are solutions of

$$\frac{d^3\psi}{dv^3} + V_2\frac{d\psi}{dv} + V_3\psi = 0, \tag{41}$$

and $\psi^{(4)}$ is a solution of

$$\frac{d^3\psi}{dv^3} + V_2\frac{d\psi}{dv} + V_3\psi = 1. \tag{42}$$

We denote the three Wronskians of the $\psi^{(k)}$ by $V^{(k)}$, so that, for instance,

$$V^{(1)} = \psi^{(2)}\psi^{(3)}_v - \psi^{(3)}\psi^{(2)}_v.$$

Then by variation of parameters we find

$$\psi^{(4)} = \sum_{k=1}^{3}\left(\int V^{(k)}dv + c^{(k)}\right)\psi^{(k)}. \tag{43}$$

Equation (41) should be compared with equation (31) which is satisfied by G, while the $V^{(k)}$ satisfy the Lagrange adjoint of (41), namely

$$\frac{d^3V}{dv^3} + V_2\frac{dV}{dv} + (V_2' - V_3)V = 0. \tag{44}$$

This equation is to be compared with (32), which is satisfied by F.

We are able to express F and G in terms of the functions $\varphi^{(k)}$ and $\psi^{(k)}$. In fact, F is a solution of (16) and (32), while G is a solution of (29) and (31). Now, reference to (37), (39), (41), and (44) shows that we have

$$F = \sum_{k=1}^{3} V^{(k)}\varphi^{(k)}, \qquad G = \sum_{k=1}^{3} U^{(k)}\psi^{(k)}, \tag{45}$$

where the $V^{(k)}$ and $U^{(k)}$ are the Wronskians previously defined.

Recalling then that $f = -\log F$, $g = -\log G$ we can show that the values obtained for y by both methods of integrating system (Δ) reduce to the following:

$$\begin{aligned} Fy^{(1)} &= U^{(3)}V^{(2)} - U^{(2)}V^{(3)}, \\ Fy^{(2)} &= U^{(1)}V^{(3)} - U^{(3)}V^{(1)}, \\ Fy^{(3)} &= U^{(2)}V^{(1)} - U^{(1)}V^{(2)}, \\ 1y^{(4)} &= \sum_{k=1}^{3}\left(\int U^{(k)}du + \int V^{(k)}dv + c^{(k)}\right)y^{(k)}. \end{aligned} \tag{46}$$

The values for z, obtained by both methods, reduce to

$$Fz^{(k)} = \varphi^{(k)}, \qquad (k = 1, 2, 3),$$

$$1z^{(4)} = \sum_{k=1}^{3}\left(\int U^{(k)}du + \int V^{(k)}dv + c^{(k)}\right)z^{(k)},$$

while the values for ρ reduce to

$$G\rho^{(k)} = \psi^{(k)}, \qquad (k = 1, 2, 3),$$

$$1\rho^{(4)} = \sum_{k=1}^{3}\left(\int U^{(k)}du + \int V^{(k)}dv + c^{(k)}\right)\rho^{(k)}.$$

VII. Geometrical Applications.

We shall now proceed to apply formulas (46), which were obtained by integrating system (Δ). These formulas give the coördinates of an arbitrary point P_y on the most general surface S_y which has the property that the axis curves of the parametric conjugate net are indeterminate.

Let us first deduce the equation of the plane tangent to S_y at P_y. This plane is determined by the three points y, y_u, and y_v, and when its equation is found in the ordinary way, it turns out to be

$$x_4 = \sum_{k=1}^{3}\left(\int U^{(k)}du + \int V^{(k)}dv + c^{(k)}x_k\right). \tag{47}$$

This equation has the same form as the last of (46). As is well known, this plane osculates at P_ρ the curve $u =$ const. through P_ρ on the cone S_ρ, and also osculates at P_z the curve $v =$ const. through P_z on the cone S_z. The two families of curves, $u =$ const. on $S\rho$, and $v =$ const. on S_z, may then be said to have the same set of osculating planes, namely the set of tangent planes of S_y.

If we differentiate equation (47) with respect to u, we obtain

$$\sum_{k=1}^{3} U^{(k)}x_k = 0. \tag{48}$$

This plane cuts the surface S_y in the curve $u =$ const. which passes through P_y, and moreover is tangent to the cone S_z along the corresponding generator $u =$ const. In the same way, we obtain by differentiating equation (47) with respect to v,

$$\sum_{k=1}^{3} V^{(k)}x_k = 0. \tag{49}$$

This plane cuts S_y in the curve $v =$ const. which passes through P_y, and is tangent to the cone S_ρ along a generator $v =$ const.

The functions $U^{(k)}$ are the coördinates of the enveloping plane of the cone S_z, and the $V^{(k)}$ are the coördinates of the enveloping plane of S_ρ. Reference to equations (46) shows that the coördinates of P_y depend only on these six functions and the three arbitrary constants $c^{(k)}$. So then, except for these three constants, the surface S_y is determined when the cones S_z and S_ρ are given.

Let us seek for a geometrical meaning for the constants $c^{(k)}$. To this end, let us consider the following projective transformation:

$$
\begin{aligned}
\omega x_1' &= x_1 \quad , \\
\omega x_2' &= x_2 \quad , \\
\omega x_3' &= x_3 \quad , \\
\omega x_4' &= a^{(1)}x_1 + a^{(2)}x_2 + a^{(3)}x_3 + x_4 .
\end{aligned} \tag{50}
$$

This transformation is completely characterized by the property of leaving invariant the point (0, 0, 0, 1) and every stright line through this point. It therefore leaves P_σ and the cones S_z and S_ρ invariant. But it changes the surface S_y, so that the first three coördinates of P_y remain the same as given by (46), while the fourth becomes

$$
y^{(4)} = \sum_{k=1}^{3} \left(\int U^{(k)} du + \int V^{(k)} dv + c^{(k)} + a^{(k)} \right) y^{(k)} .
$$

Therefore assigning various values to the constants $c^{(k)}$ merely amounts to making projective transformations of the simple type (50). In other words, the surface S_y is determined by the cones S_z and S_ρ up to a projective transformation of this type.

We may now summarize our results. *Let there be given two cones with a common vertex. Then there exists a three-parameter family of surfaces, each surface having the following properties*:

1. *The tangent planes of the two cones cut the surface in a conjugate system.*
2. *The axis curves of this system are indeterminate.*
3. *The axis of every point on the surface, defined with reference to this system, passes through the common vertex of the given cones. Moreover, the family of surfaces has the property that any surface of it may be obtained from any other surface of it by means of the projective transformation that leaves invariant the common vertex of the two given cones and also every line through this point.*

VIII. A Special Case.

We have assumed throughout this work that $g_{uv}f_{uv} \neq 0$. The fifth integrability condition, when $c' = 0$ and S_y is nondegenerate, shows that

$f_{uv} = 0$ if, and only if, S_z degenerates into a curve. This curve would be a straight line, since S_z is also a developable. By analogy, $g_{uv} = 0$ if, and only if, S_ρ reduces to a straight line. Now it happens here, as it often does happen, that the essential results of the investigation are true independently of certain restricting hypotheses used in deriving them. For example, equations (46) are valid, even if one or both of the two cones do reduce to straight lines.

Let us now specialize the six functions and three constants appearing in (46) so as to obtain a very simple surface which has on it a conjugate system with indeterminate axis curves. In the last of equations (46) we shall take the lower limit of integration to be zero both for u and for v. Then we shall take

$$\begin{array}{lll} U^{(1)} = -1, & U^{(2)} = 0, & U^{(3)} = u, \\ V^{(1)} = 0, & V^{(2)} = -1, & V^{(3)} = v, \\ c^{(1)} = 0, & c^{(2)} = 0, & c^{(3)} = -\tfrac{1}{2}. \end{array}$$

In this way equations (46) reduce to

$$\begin{gathered} Fy^{(1)} = -u, \qquad Fy^{(2)} = -v, \qquad Fy^{(4)} = -1, \\ Fy^{(4)} = \tfrac{1}{2}(u^2 + v^2 + 1). \end{gathered} \tag{51}$$

Let us introduce nonhomogeneous coördinates by the equations

$$x = y^{(1)}/y^{(4)}, \qquad y = y^{(2)}/y^{(4)}, \qquad z = y^{(3)}/y^{(4)}.$$

Then from (51) we obtain

$$x = -\frac{2u}{u^2+v^2+1}, \qquad y = -\frac{2v}{u^2+v^2+1}, \qquad z = -\frac{2}{u^2+v^2+1}. \tag{52}$$

Eliminating the parameters u and v, we obtain

$$x^2 + y^2 + (z+1)^2 = 1. \tag{53}$$

At the same time equations (48) and (49) reduce to

$$x = uz, \qquad y = vz.$$

Therefore, the unit sphere, tangent to the xy-plane at the origin, is a surface on which there exists a conjugate system with indeterminate axis curves. The pencil of planes through the x-axis cuts the sphere in the curves $v = \text{const.}$, and the pencil of planes through the y-axis cuts the sphere in the curves $u = \text{const.}$ The cones S_z and S_ρ are simply these two coordinate axes. And the axis of every point on the sphere passes through the origin.

The generalization to quadric surfaces is immediate. *At any point P on any quadric surface select any two tangent lines separating harmonically the two generators through P. With each of these tangents as axis, construct a pencil of planes. The planes of these two pencils cut the quadric in two one-parameter families of plane curves which constitute a conjugate system with indeterminate axis curves. The axis of every point on the quadric, defined with reference to this system, passes through P.*

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